IFT 4030/7030, Machine Learning for Signal Processing

Week3: Signal Processing Primer

Cem Subakan

Admin

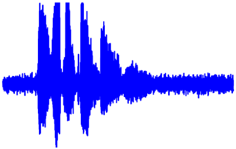
- How were the first and second labs?
  - Comment était le labs 1, 2?
- The next lab is on friday!
  - On aura le deuxième lab en vendredi.
- How are the project proposals coming along? The proposals are due the second week of october.
  - Comment va la réflection sur les projets? Les propositions de projets sont dûs la deuxième semaine de l'octobre!

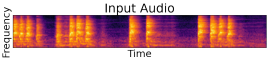
Today

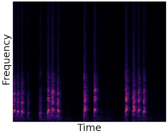
- Signals/Time series, what are they?
  - Qu'est-ce qu'on comprend quand on dit 'signaux/séries temporelles'?

Today

- Signals/Time series, what are they?
  - Qu'est-ce qu'on comprend quand on dit 'signaux/séries temporelles'?
- How do we represent signals?
  - Comment est-ce qu'on répresente les signaux?
    - Time Domain / Frequency Domain / Time+Frequency Domain

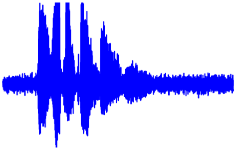


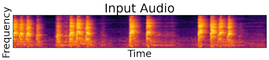


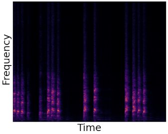


Today

- Signals/Time series, what are they?
  - Qu'est-ce qu'on comprend quand on dit 'signaux/séries temporelles'?
- How do we represent signals?
  - Comment est-ce qu'on répresente les signaux?
    - Time Domain / Frequency Domain / Time+Frequency Domain



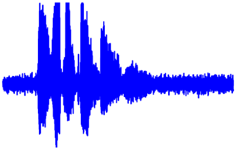


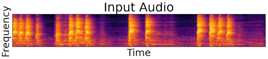


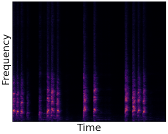
- The Fourier Transform
  - The Short-Time Fourier Transform

Today

- Signals/Time series, what are they?
  - Qu'est-ce qu'on comprend quand on dit 'signaux/séries temporelles'?
- How do we represent signals?
  - Comment est-ce qu'on répresente les signaux?
    - Time Domain / Frequency Domain / Time+Frequency Domain



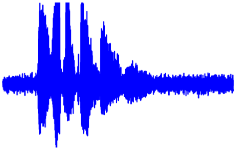


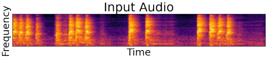


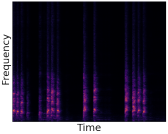
- The Fourier Transform
  - The Short-Time Fourier Transform
- Filtering / Convolution
  - Convolution

Today

- Signals/Time series, what are they?
  - Qu'est-ce qu'on comprend quand on dit 'signaux/séries temporelles' ?
- How do we represent signals?
  - Comment est-ce qu'on répresente les signaux?
    - Time Domain / Frequency Domain / Time+Frequency Domain



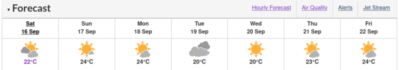


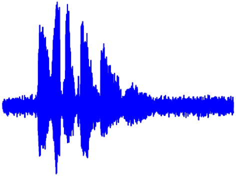


- The Fourier Transform
  - The Short-Time Fourier Transform
- Filtering / Convolution
  - Convolution
- Sampling
  - Échantillonnage

Signals

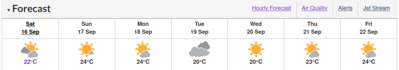
- A dry definition: A signal is an ordered collection of numbers
  - Une définition sec: Un signal est une collection des chiffres en ordre.





Signals

- A dry definition: A signal is an ordered collection of numbers
  - Une définition sec: Un signal est une collection des chiffres en ordre.



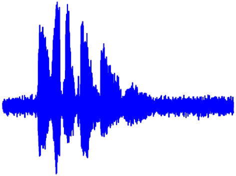


Table of Contents

Representing Signals/Time Series

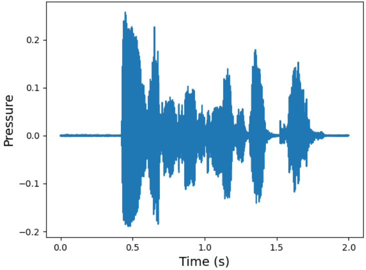
- There are different ways of representing time series/signals.
  - Il existe plusieurs façons de répresenter les signaux.
- Each representation type has its pros / cons.
  - Chaque type de représentation a leurs avantages / desavantages.

Representing Signals/Time Series

- There are different ways of representing time series/signals.
  - Il existe plusieurs façons de répresenter les signaux.
- Each representation type has its pros / cons.
  - Chaque type de représentation a leurs avantages / desavantages.
- Some typical options: Time Domain, Frequency Domain, Time+Frequency Domain

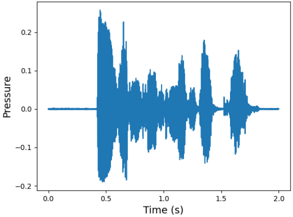
Sound as Signal

- We will start with sounds as an example, but any time series would do.
  - On commencera avec les sons comme un exemple, mais n'importe quel time series serait ok.
- Let's `listen`.



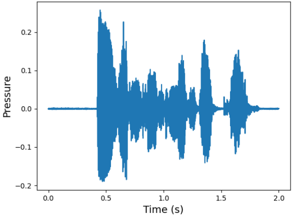
The time domain

- It's kinda difficult what to take out of this by looking at it.
  - Je sais pas quoi faire en regardant ça.



The time domain

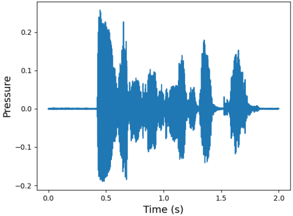
- It's kinda difficult what to take out of this by looking at it.
  - Je sais pas quoi faire en regardant ça.



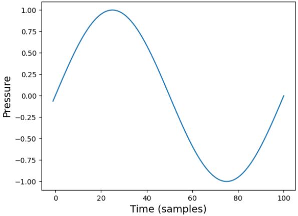
- It's possible however to express this in terms of simpler waveforms.
  - C'est possible par contre de l'exprimer ce waveform en terme des waveform plus simples.

The time domain

- It's kinda difficult what to take out of this by looking at it.
  - Je sais pas quoi faire en regardant ça.

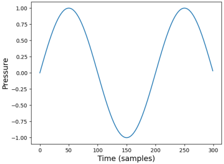


- It's possible however to express this in terms of simpler waveforms.
  - C'est possible par contre de l'exprimer ce waveform en terme des waveform plus simples.
- Sinusoids!! (They are special)

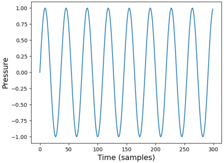


Sinusoids

- 80 Hz, `listen`.



- 440 Hz, `listen`. This is how we tune our instruments.
  - On tune nos instruments à La 440Hz.

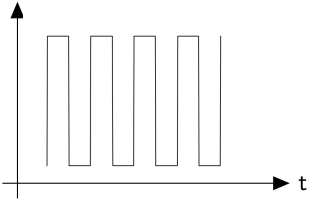


- 880 Hz, `listen`
- 1760 Hz, `listen`
- 15000 Hz, `are you able to hear this at all?`
  - Pouvez-vous entendre ceci?

Table of Contents

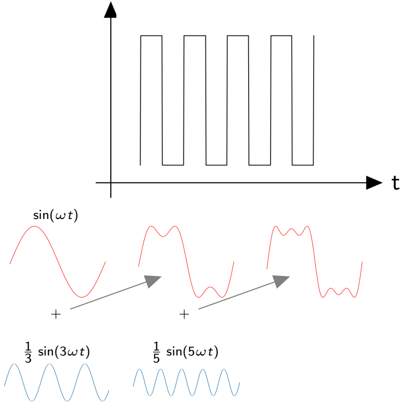
Decomposing a signal

- Let's decompose a square wave with sines.
  - Decomposons un square wave.



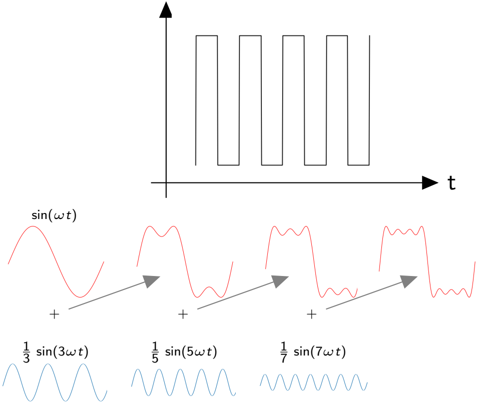
Decomposing a signal

- Let's decompose a square wave with sines.
  - Decomposons un square wave.



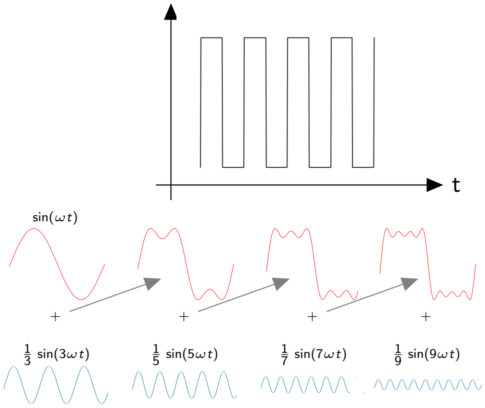
Decomposing a signal

- Let's decompose a square wave with sines.
  - Decomposons un square wave.



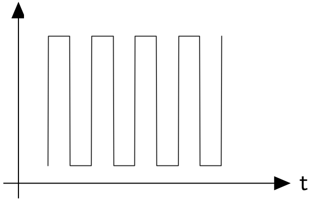
Decomposing a signal

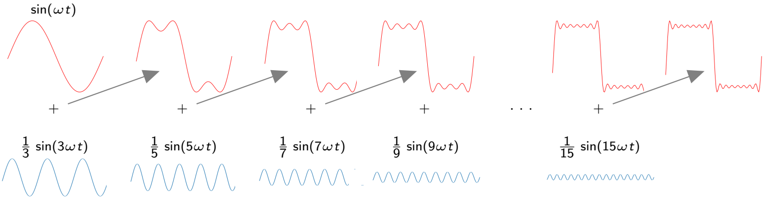
- Let's decompose a square wave with sines.
  - Decomposons un square wave.



Decomposing a signal

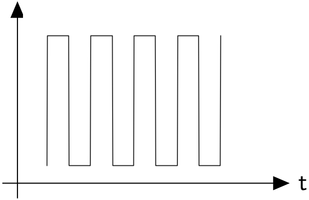
- Let's decompose a square wave with sines.
  - Decomposons un square wave.

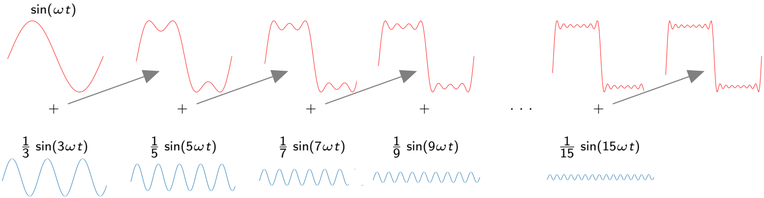




Decomposing a signal

- Let's decompose a square wave with sines.
  - Decomposons un square wave.





- So we can approximate a square wave with
  - Alors on peut approximer un onde carré avec

$$\mathrm{SW}(\omega t) \approx \sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \ldots$$

Frequency Representation

- The goal is to find the contribution of each sinusoid (or ‘frequency’).
  - Le but est d’obtenir la contribution de chaque sinusoid (ou ‘fréquence).

Frequency Representation

- The goal is to find the contribution of each sinusoid (or ‘frequency’).
  - Le but est d’obtenir la contribution de chaque sinusoid (ou ‘fréquence).
- We can calculate similarity between different sinusoids and the input waveform.
  - On peut calculer des similarités entre differents sinusoids et l’entrée.

Frequency Representation

- The goal is to find the contribution of each sinusoid (or ‘frequency’).
  - Le but est d’obtenir la contribution de chaque sinusoid (ou ‘fréquence).
- We can calculate similarity between different sinusoids and the input waveform.
  - On peut calculer des similarités entre differents sinusoids et l’entrée.
- Do you remember how we calculate the similarity between two vectors?
  - Est-ce que vous vous en souvenez comment-t-on calcule similarité entre deux vecteurs?

Frequency Representation

- The goal is to find the contribution of each sinusoid (or ‘frequency’).
  - Le but est d’obtenir la contribution de chaque sinusoid (ou ‘fréquence).
- We can calculate similarity between different sinusoids and the input waveform.
  - On peut calculer des similarités entre differents sinusoids et l’entrée.
- Do you remember how we calculate the similarity between two vectors?
  - Est-ce que vous vous en souvenez comment-t-on calcule similarité entre deux vecteurs?

- **Inner Product!**

Table of Contents

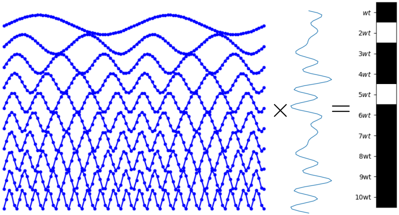
Fourier Transform

- Fourier Transform is basically a projection over sinusoids.
  - Dans le fond, FT est une projection sur des sinusoids.
- Let's do that for this signal.
  - Faisons ça pour ce signal.

Fourier Transform

- Fourier Transform is basically a projection over sinusoids.
  - Dans le fond, FT est une projection sur des sinusoids.
- Let's do that for this signal.
  - Faisons ça pour ce signal.

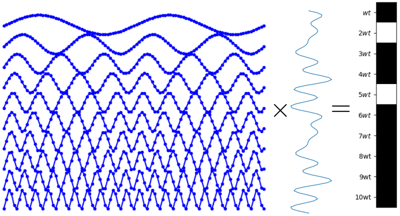
- FT in principle:



Fourier Transform

- Fourier Transform is basically a projection over sinusoids.
  - Dans le fond, FT est une projection sur des sinusoids.
- Let's do that for this signal.
  - Faisons ça pour ce signal.

- FT in principle:



- This is nice, but is this general enough?
  - Bon, mais est-ce assez générale?

No! You need cosines to span the space

- The signal we saw before was,
  - Le signal qu'on avait vu était:

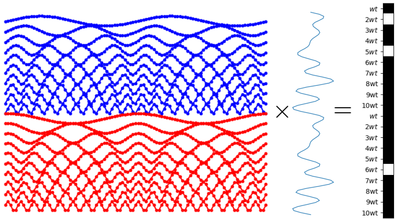
$$y(t) = \sin(2\omega t) + \sin(5\omega t) + \underbrace{\cos(6\omega t)}_{\text{We missed this guy!}}$$

No! You need cosines to span the space

- The signal we saw before was,
  - Le signal qu'on avait vu était:

$$y(t) = \sin(2\omega t) + \sin(5\omega t) + \underbrace{\cos(6\omega t)}_{\text{We missed this guy!}}$$

- Let's extend our bases,
  - Élargons notre ensemble de bases:

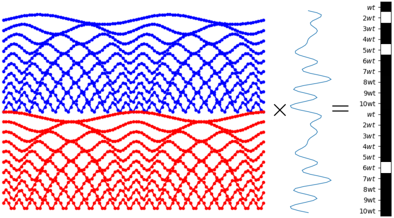


No! You need cosines to span the space

- The signal we saw before was,
  - Le signal qu'on avait vu était:

$$y(t) = \sin(2\omega t) + \sin(5\omega t) + \underbrace{\cos(6\omega t)}_{\text{We missed this guy!}}$$

- Let's extend our bases,
  - Élargons notre ensemble de bases:



- But wait, I remember that Fourier Transform gave us complex numbers. What's that about?
  - Mais chuis confus-là, je me souviens que FT nous donnait des chiffres complexes?

Fourier Transform Formal Definition

- Remember Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- So it seems like, we can use complex exponentials to project on to cosines and sines at the same time.
  - On alors peut utiliser les exponentiels complexes pour projeter sur les cosines et sines en meme temps.

Fourier Transform Formal Definition

- Remember Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- So it seems like, we can use complex exponentials to project on to cosines and sines at the same time.
  - On alors peut utiliser les exponentiels complexes pour projeter sur les cosines et sines en meme temps.
- Discrete Fourier Transform (DFT):

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-j2\pi \frac{k}{N} n\right), \text{ where}$$
$$\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}],\ k \in \{0, \ldots, N-1\}.$$

- Note that k corresponds to frequencies, and we have the same number of frequencies as the signal length N.
  - Notez que k indice est la fréquence, et on a meme nombres de fréquences que le longeur N.

Fourier Transform Formal Definition

- Remember Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

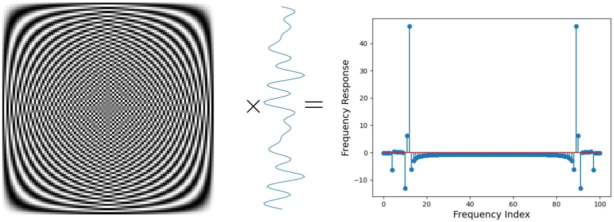
- So it seems like, we can use complex exponentials to project on to cosines and sines at the same time.
  - On alors peut utiliser les exponentiels complexes pour projeter sur les cosines et sines en meme temps.
- Discrete Fourier Transform (DFT):

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-j2\pi\frac{k}{N}n\right), \text{ where}$$
$$\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}],\ k \in \{0, \ldots, N-1\}.$$

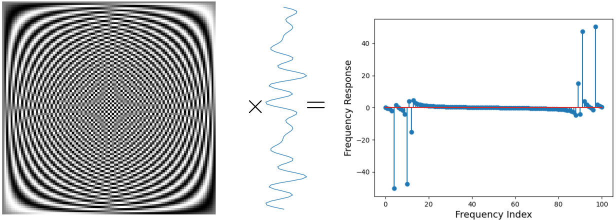
- Note that k corresponds to frequencies, and we have the same number of frequencies as the signal length N.
  - Notez que k indice est la fréquence, et on a meme nombres de fréquences que le longeur N.
- By the way, do you see that this is a matrix? $\exp\left(-j2\pi\frac{k}{N}n\right)$
  - Vous voyez que c'est une matrice?

DFT in action

- Real Part / La partie réel



- Imaginary Part / La partie imaginaire



Remarks

- Notice that les results are symmetric. This is because of the construction of the DFT matrix.
  - Notez que les résultats sont symmetriques. C'est à cause de la construction de la matrice DFT.

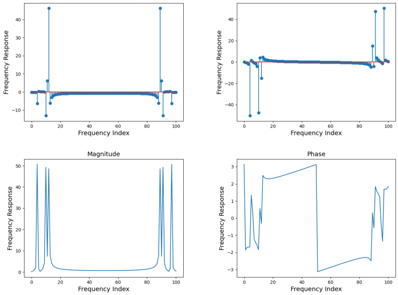
Remarks

- Notice that les results are symmetric. This is because of the construction of the DFT matrix.
  - Notez que les résultats sont symmetriques. C'est à cause de la construction de la matrice DFT.
- Note that the DFT matrix repeats after the half. We will get back to this later.
  - Notez que les matrices DFT répetent après la moitié. On va parler de ça après.

Another way of interpreting DFT

- Note that we get complex numbers. We can calculate magnitude and phase.
  - Notez qu'on obtiens des nombres complex. Donc on peut calculer la magnitude et la phase.

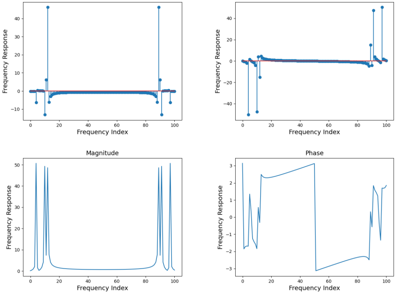
$$
\begin{aligned}
|X_k| &= \sqrt{\mathrm{Re}(X_k)^2 + \mathrm{Im}(X_k)^2} \\
\angle X_k &= \mathrm{atan2}(\mathrm{Im}(X_k), \mathrm{Re}(X_k))
\end{aligned}
$$



Another way of interpreting DFT

- Note that we get complex numbers. We can calculate magnitude and phase.
  - Notez qu'on obtiens des nombres complex. Donc on peut calculer la magnitude et la phase.

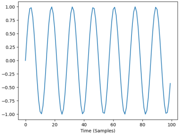
$$
\begin{aligned}
|X_k| &= \sqrt{\mathrm{Re}(X_k)^2 + \mathrm{Im}(X_k)^2} \\
\angle X_k &= \mathrm{atan2}(\mathrm{Im}(X_k), \mathrm{Re}(X_k))
\end{aligned}
$$

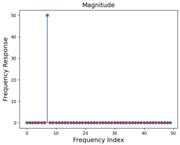


- Magnitude is easy to interpret. Phase is not as easy always.
  - Magnitude est facile est interpreter. La phase n'est pas toujours si facile à interpreter.

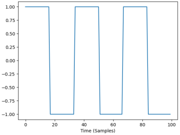
DFTs of different signals

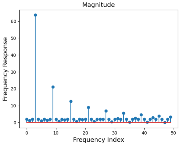
- A single sinusoid





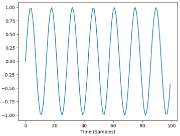
- Square wave

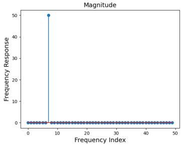




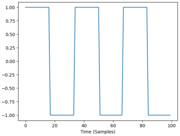
DFTs of different signals

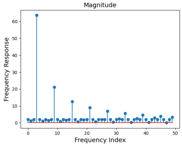
- A single sinusoid



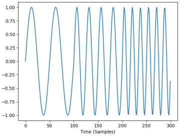


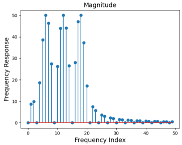
- Square wave





- Sinusoids with increasing frequencies



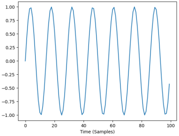


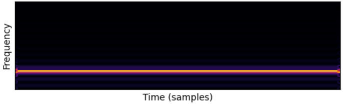
- So, for time varying signals, DFT is not that great.
  - Pour les signaux qui varie avec le temps DFT n'est pas idéal.

Table of Contents

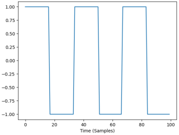
Time-frequency representation

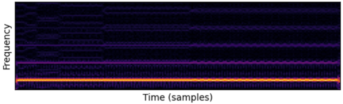
- A single sinusoid





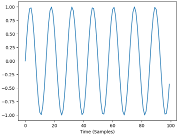
- Square wave

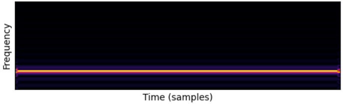




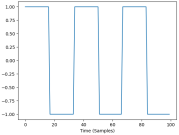
Time-frequency representation

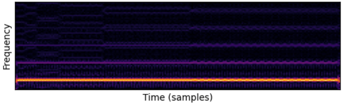
- A single sinusoid



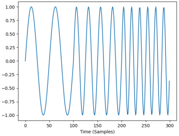


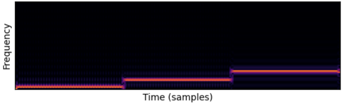
- Square wave





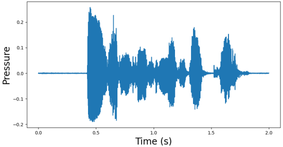
- Sinusoids with increasing frequencies





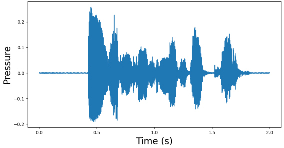
A real example for speech

Time Domain

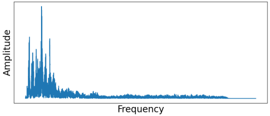


A real example for speech

Time Domain

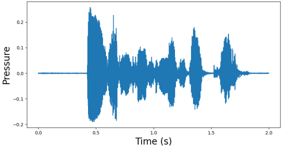


Frequency Domain – but what is this really? I see low freqs, but where is time information? – On voit freq basses mais pas d'info sur temps.

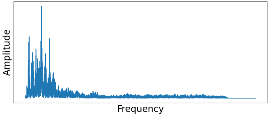


A real example for speech

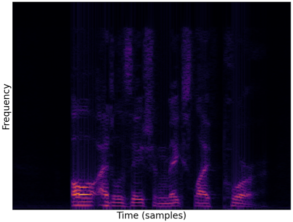
Time Domain



Frequency Domain – but what is this really? I see low freqs, but where is time information? – On voit freq basses mais pas d'info sur temps.

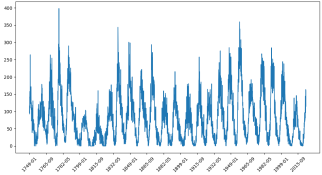


Time-Frequency Domain – We 'see' the signal. – on voit le signal.



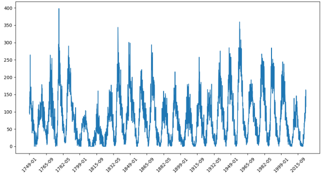
An example from another domain

Time Domain
Counts of sunspots wrt. time

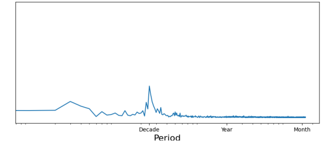


An example from another domain

Time Domain
Counts of sunspots wrt. time

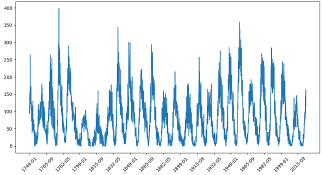


Frequency Domain

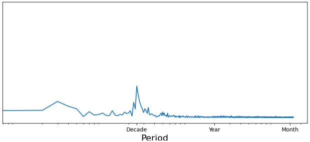


An example from another domain

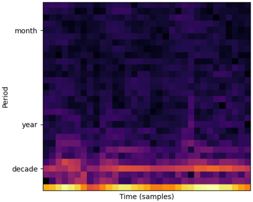
Time Domain
Counts of sunspots wrt. time



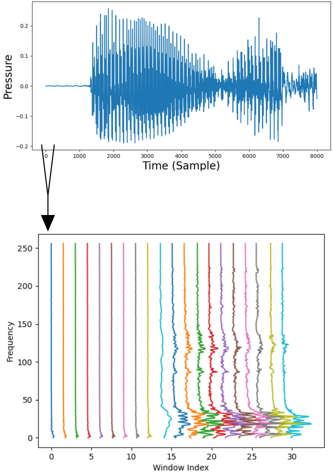
Frequency Domain



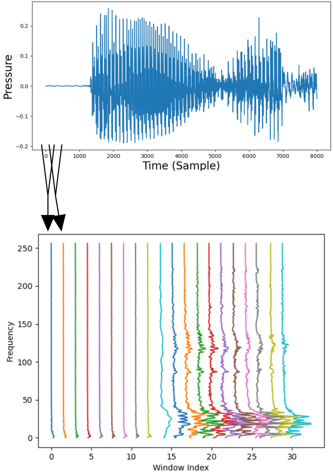
Time-Frequency Domain



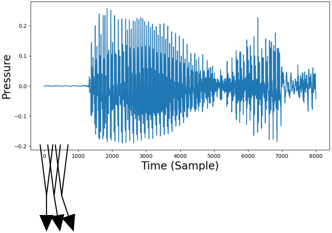
How to Calculate the Spectrogram (STFT)

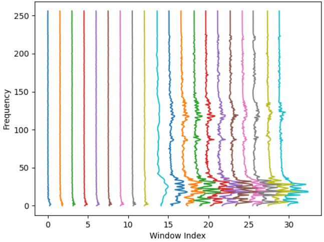


How to Calculate the Spectrogram (STFT)

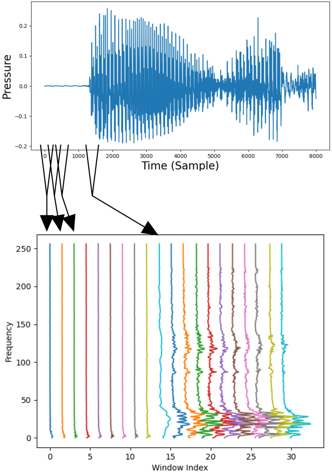


How to Calculate the Spectrogram (STFT)

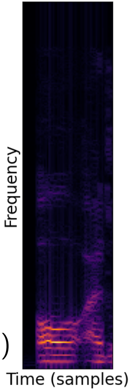




How to Calculate the Spectrogram (STFT)

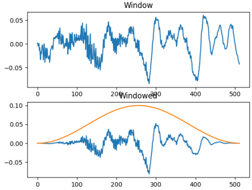


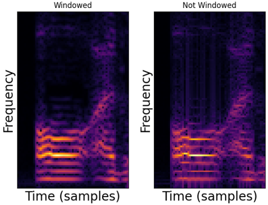
It's nice to show it as an image! (imshow())



STFT considerations

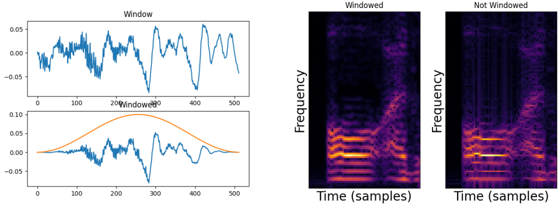
- Windowing (Utilisation de fenetres)
  - We apply a window, before calculating the DFT.
    - On applique une fenetre avant de calculer le DFT.





STFT considerations

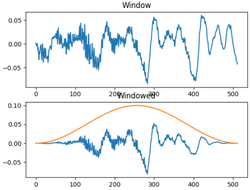
- Windowing (Utilisation de fenetres)
  - We apply a window, before calculating the DFT.
    - On applique une fenetre avant de calculer le DFT.

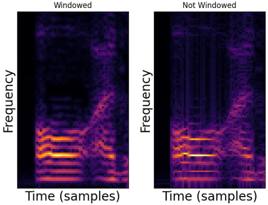


- Notice that not windowed STFT contains more high frequency artifacts
  - Notez que STFT qui n'est pas windowé contiens plus des artifacts de haute fréquence.

STFT considerations

- Windowing (Utilisation de fenetres)
  - We apply a window, before calculating the DFT.
    - On applique une fenetre avant de calculer le DFT.

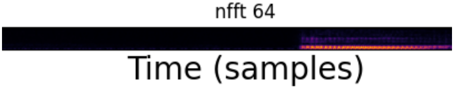




- Notice that not windowed STFT contains more high frequency artifacts
  - Notez que STFT qui n'est pas windowé contiens plus des artifacts de haute fréquence.
- We add an overlap to make up for using tapering windows. / On ajoute un overlap aussi pour compenser pour les fenetres qui réduit en amplitude.

Time/Frequency Tradeoff

- Heisenberg's uncertainty principle - We can not know the frequency and time location of a wave.
  - On ne peut pas déterminer la fréquence et et la localisation temporelle d'une onde.
- In the context of spectrograms: Big DFT sacrifice temporal resolution, Small DFTs have bad frequency resolution
  - Grand DFT a une mauvaise resolution temporelle, Petit DFTs ont une mauvaise resolution fréquentielle



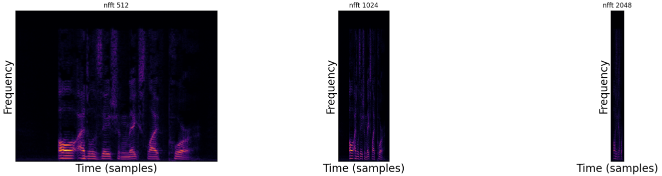
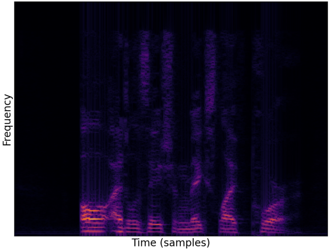


Table of Contents

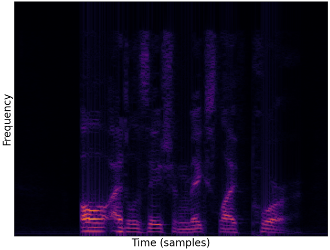
Mel-Frequency Spectrograms

- You notice that most of the energy is concentrated on the lower frequencies.
  - Vous vous rendez compte que l'energie est plutot concentrée dans les frequences bas.



Mel-Frequency Spectrograms

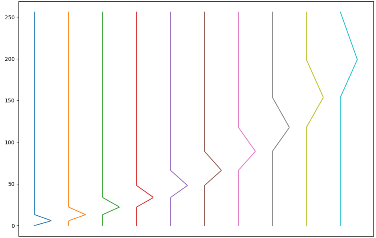
- You notice that most of the energy is concentrated on the lower frequencies.
  - Vous vous rendez compte que l'energie est plutot concentrée dans les frequences bas.



- This is a bit wasteful.

Mel-Frequency Spectrograms

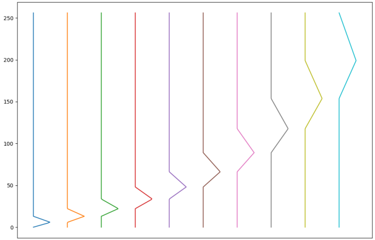
- We can 'warp' the frequency axis.



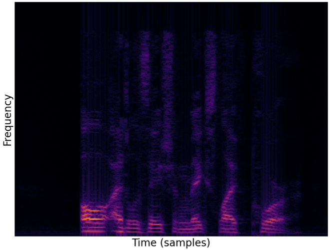
Mel-Frequency Spectrograms

- We can ‘warp’ the frequency axis.

$\sum_f$



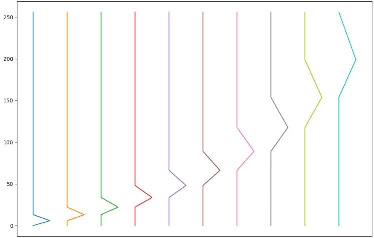
$\odot$



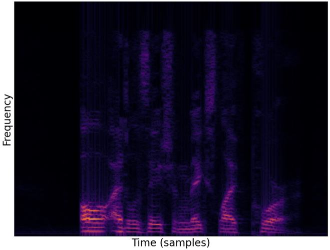
Mel-Frequency Spectrograms

- We can ‘warp’ the frequency axis.

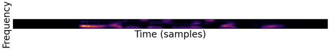
$\sum_f$



$\odot$

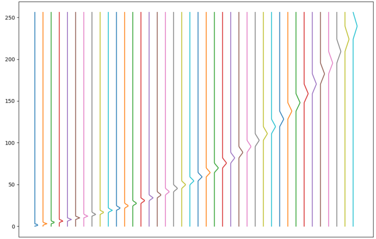


$=$



Mel-Frequency Spectrograms

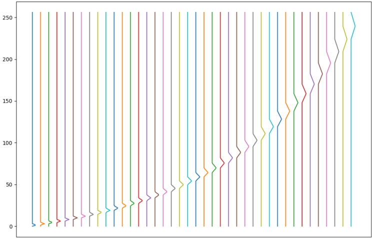
- We can ‘warp’ the frequency axis with more filters also (e.g. 40).



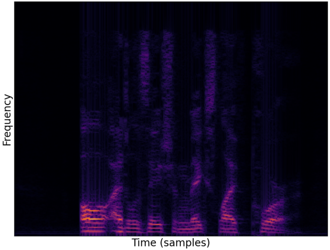
Mel-Frequency Spectrograms

- We can ‘warp’ the frequency axis with more filters also (e.g. 40).

$\sum_f$



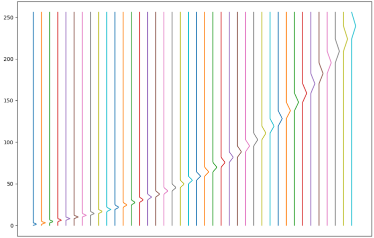
$\odot$



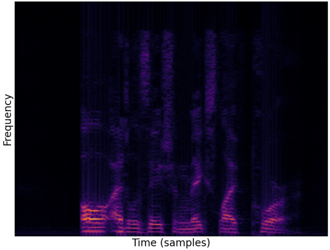
Mel-Frequency Spectrograms

- We can 'warp' the frequency axis with more filters also (e.g. 40).

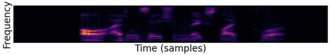
$\sum_f$



$\odot$

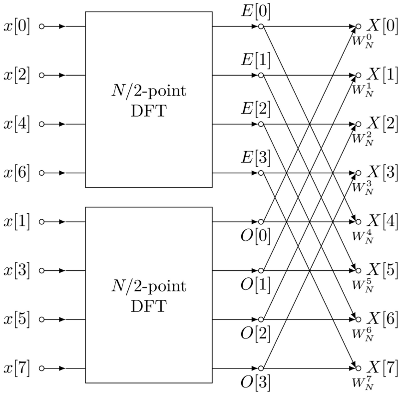


$=$



Fast Fourier Transform (FFT)

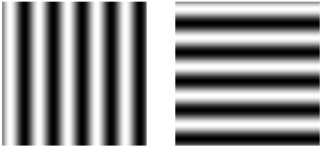
- DFT matrix has symmetries.
- DFT can be decomposed into DFTs of half the size.
  - La DFT peut etre decomposé aux 2 DFTs avec la moitié de la taille originale.
- We reduce the complexity from $\mathcal{O}(N^2)$(*why?*) to $\mathcal{O}(N \log N)$.



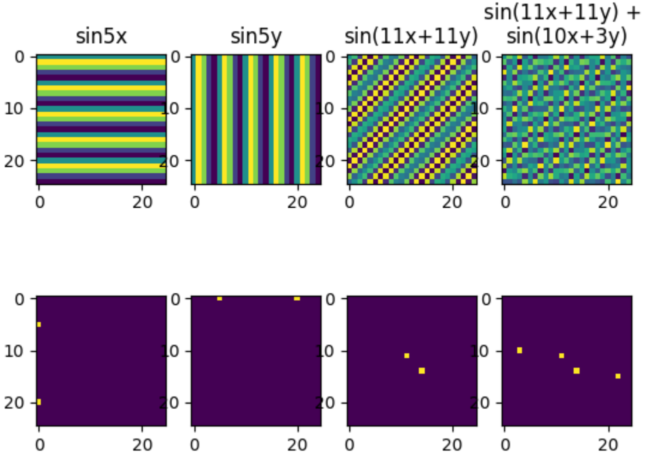
- Whenever you can use FFTs, use em!

Image DFTs

- We can généralize to images as well
  - On peut généraliser aux images aussi.
- The DFT bases in 2d
  - Les bases DFT en 2d



Some example DFTs



How to calculate 2d DFTs?

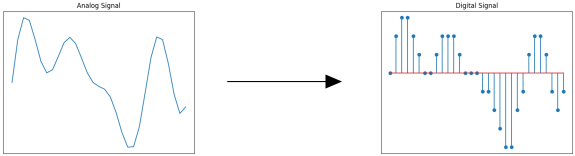
- For images $Y = FXF$
- For Tensors $F_{il_1} X_{ijk} F_{jl_2} F_{kl_3} \rightarrow Y_{l_1,l_2,l_3}$

Table of Contents

Table of Contents

Analog to Digital

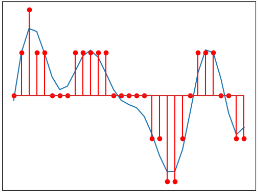
- How do we store / acquire signals?
  - Comment est-ce qu'on mets les signaux dans les ordinateurs?
- We convert the analog signals to digital.
  - On convertis signaux analogues au digital.



- It's not straightforward how to do this conversion.
  - C'est pas trivial comment faire cette conversion.

Signal Representation – Quantization

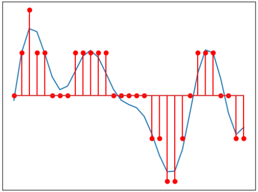
- We have to quantize a signal in order to digitize. (Quantize the y-axis)
  - On est obligé de quantiser pour digitiser. (l'axe y)



- We measure the precision in terms of bits. More bits we use, more signal-to-noise ratio we have.
  - On mésure la precision de la quantization en terme de bits. Plus de bits qu'on utilise, ça donne plus de SNR.

Signal Representation – Quantization

- We have to quantize a signal in order to digitize. (Quantize the y-axis)
  - On est obligé de quantiser pour digitiser. (l'axe y)



- We measure the precision in terms of bits. More bits we use, more signal-to-noise ratio we have.
  - On mésure la precision de la quantization en terme de bits. Plus de bits qu'on utilise, ça donne plus de SNR.
- Example average numbers:
  - 8bit - 48dB poor
  - 12 bits - 72dB okayish
  - 16 bits - 96 dB good
  - 24 bits - 144 dB overkill

Quantization in practice

8, 7, 5 bits

4, 3, 2 bits, 1 bit

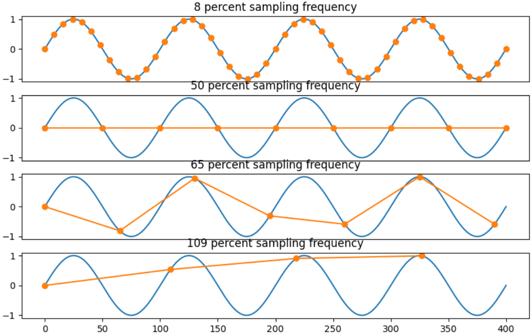
Table of Contents

Sampling / Échantillonnage

- How often we sample is also important!
  - C'est aussi important la fréquence avec la quelle on échantillon. (l'axe de temps)
    - We use Hz to measure sampling freq. / On utilize Hz pour mésurer la fréquence d'echantillonnage.
- Important: Nyquist Rate: We must sample x2 above the highest frequency we want to represent.
  - Important: Le rate de Nyquist: On doit échantillonner 2 fois au délà de la fréquence qu'on veut répresenter.
- Perceptual Limits:
  - Our ears: We hear up to 20kHz (declines with age) So above 40kHz sampling is required.
  - Seeing: We perceive only up to 60Hz, so 120 Hz or up is required.
- Limites perceptuels:
  - Nos oreilles: La limite est 20kHz. Donc on a besoin d'une fréquence d'échantillonnage de 40Hz.
  - Nos yeux: La limite est 60Hz. On a besoin donc 120Hz.
- Common sampling rates:
  - Speech: 8kHz, 16kHz, Music: 32kHz, 44.1kHz, Pro-Audio 96kHz
  - Movies: 24fps (recently 48fps) HDTV: 60fps, ...

Aliasing

- Frequencies above 50% of the sampling rate are mis-represented.
  - Les fréquences sur 50% du rate d'échantillonnage sont mal représentés.



In real-life

- Electricity consumption sampled poorly.
  - Une mauvaise échantillonnage de la consomptions de l'éléctricité.

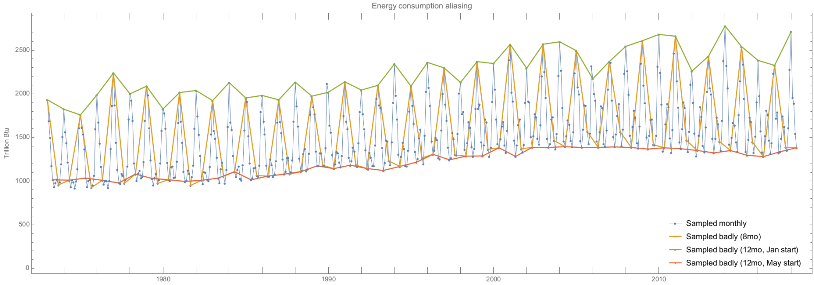


image taken from UIUC MLSP

In real-life

- Electricity consumption sampled poorly.
  - Une mauvaise échantillonnage de la consomptions de l'éléctricité.
- Different sampling leads to different conclusions.
  - Différent l'échantillonnage mène à des conclusions différentes.

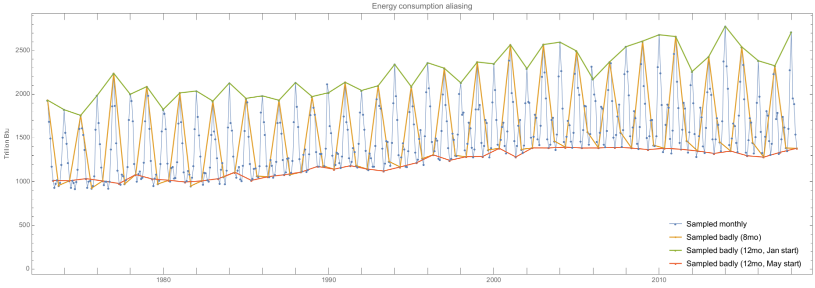
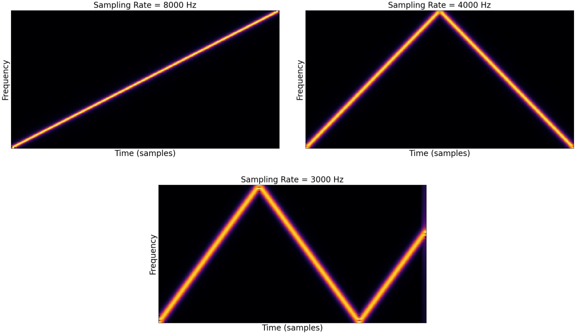


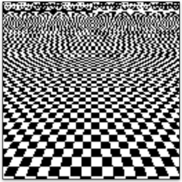
image taken from UIUC MLSP

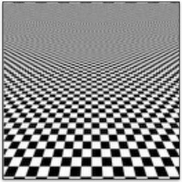
Aliasing in Sinusoid Sweeping



We have frequency sweeps from 0-4kHz, for different sampling rates.
Notice the aliasing when sampling rate is lower!
`Sweep 1` `Sweep 2` `Sweep 3`

Aliasing in Images/Video





https://www.youtube.com/watch?v=R-IVw8OKjvQ

Table of Contents

Convolution

- This is an extremely important operation that comes up all the time. (e.g. filtering)
  - C'est une opération extremement importante qu'on va voir souvent.

$$
\begin{aligned}
x(t) * w(t) := & \sum_{i=0}^{M-1} x(i)w(t-i) \\
& = x(0)w(t) + x(1)w(t-1) + x(1)w(t-2) + \dots \\
& \quad + x(M)w(t-M)
\end{aligned}
$$

Convolution

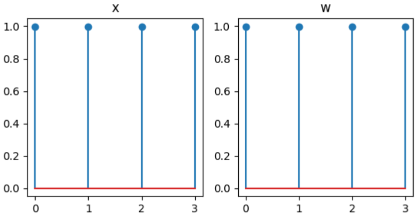
- This is an extremely important operation that comes up all the time. (e.g. filtering)
  - C'est une opération extremement importante qu'on va voir souvent.

$$
\begin{aligned}
x(t) * w(t) := \sum_{i=0}^{M-1} x(i)w(t-i) \\
= & x(0)w(t) + x(1)w(t-1) + x(1)w(t-2) + \ldots \\
& + x(M)w(t-M)
\end{aligned}
$$

- If x is of length M and w is of length N, the result is of length $M + N - 1$.
  - Si x est de longeur M et w est du longeur N, le résultat est de longeur $M + N - 1$.

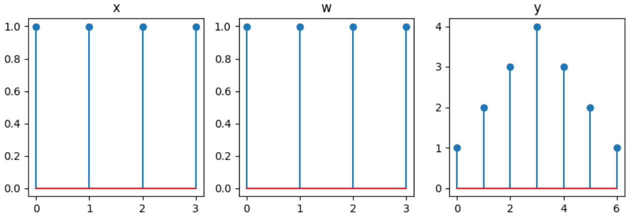
Convolution

$$y = x * w$$



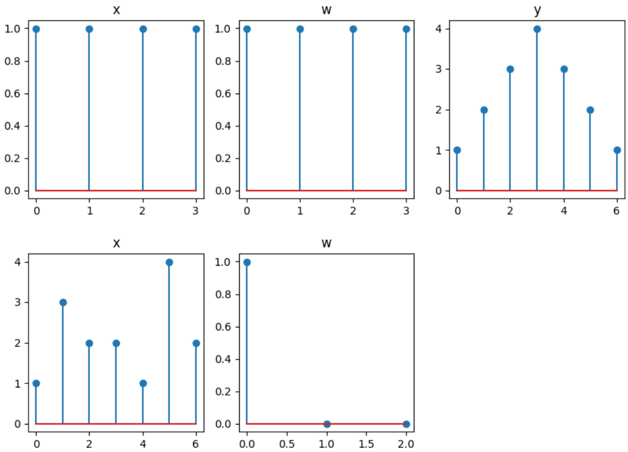
Convolution

$$y = x * w$$



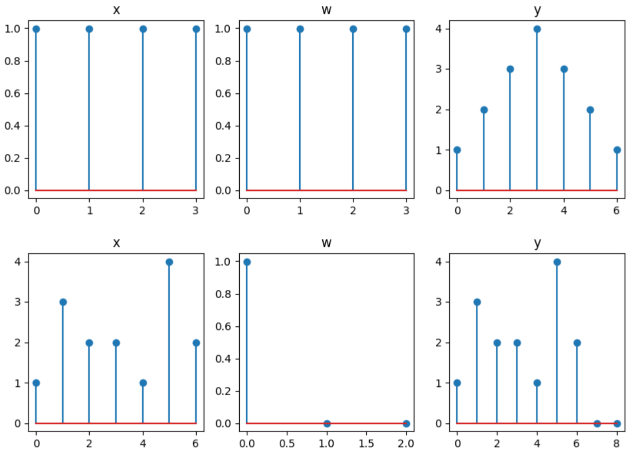
Convolution

$$y = x * w$$



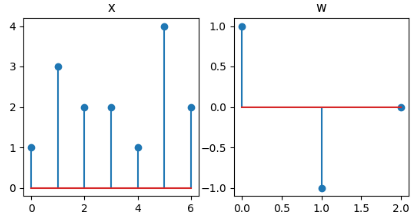
Convolution

$$y = x * w$$



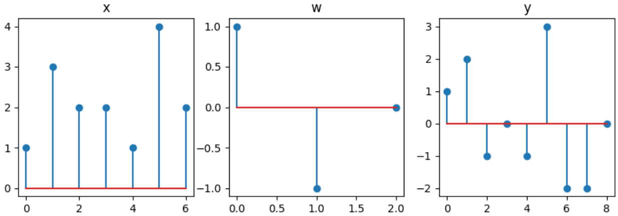
Convolution

$$y = x * w$$



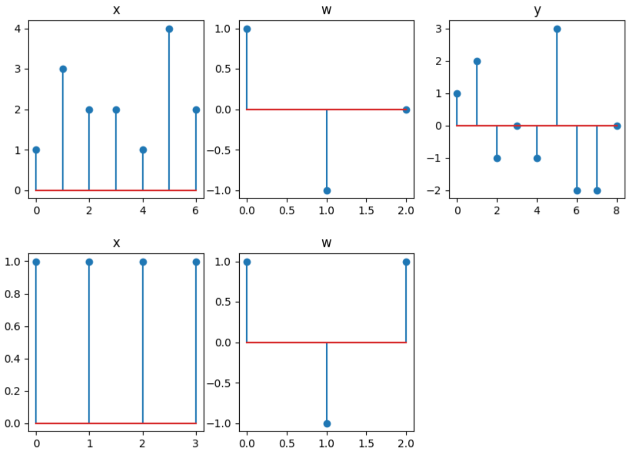
Convolution

$$y = x * w$$



Convolution

$$y = x * w$$



Convolution

$$y = x * w$$

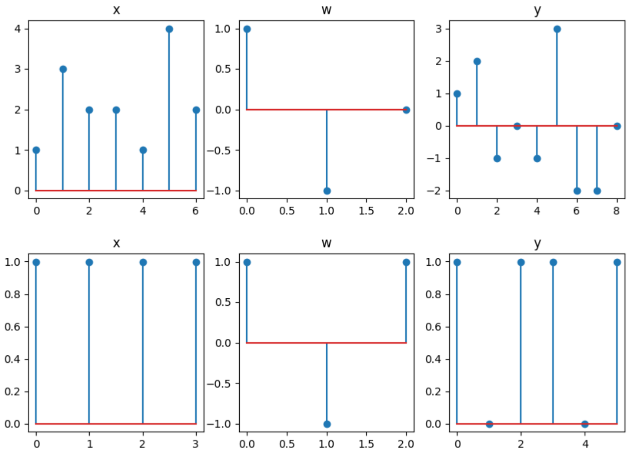


Image Convolution

$$y = x * w$$

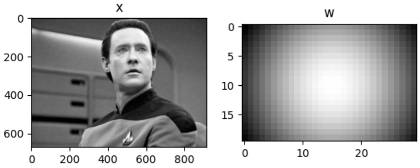


Image Convolution

$$y = x * w$$

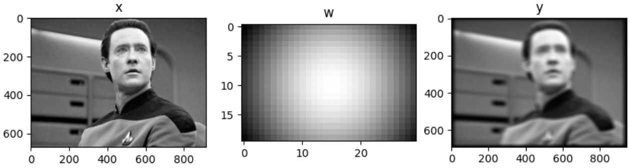


Image Convolution

$$y = x * w$$

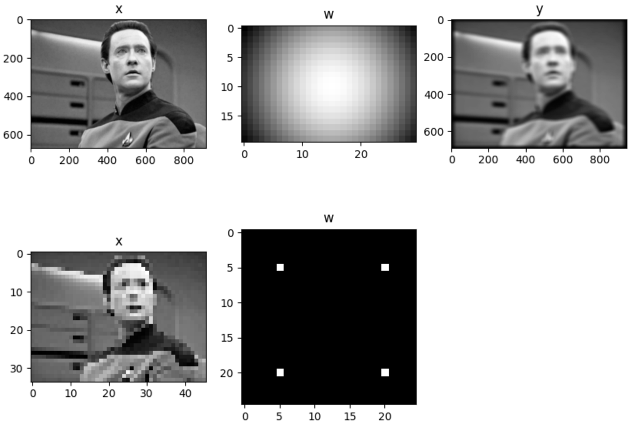
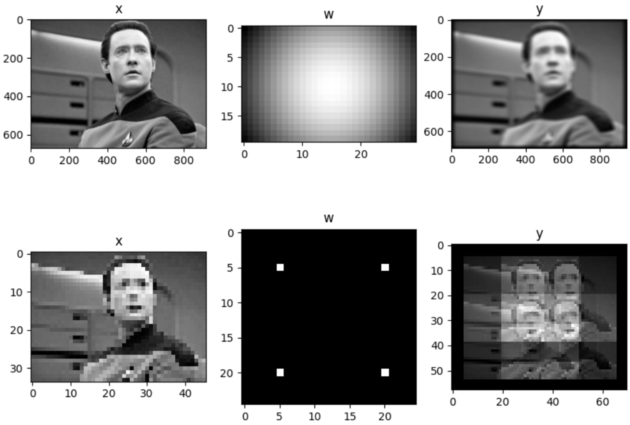


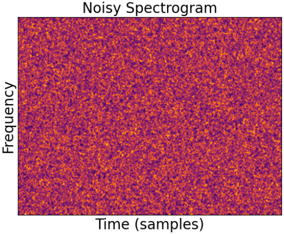
Image Convolution

$$y = x * w$$



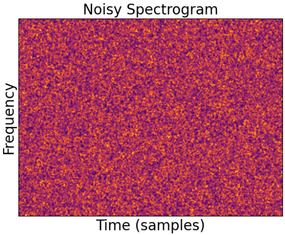
Filtering Audio with Convolution

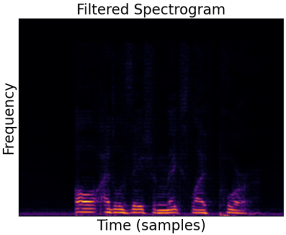
- Consider this noisy audio. `Listen`.
- The filtered version with an averaging kernel. `Listen`.
  - La version filté avec un noyau de moyenne.



Filtering Audio with Convolution

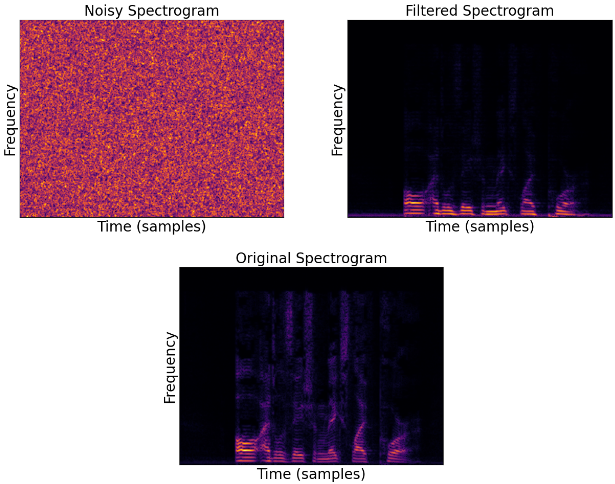
- Consider this noisy audio. `Listen`.
- The filtered version with an averaging kernel. `Listen`.
  - La version filté avec un noyau de moyenne.





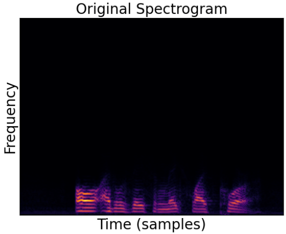
Filtering Audio with Convolution

- Consider this noisy audio. `Listen`.
- The filtered version with an averaging kernel. `Listen`.
  - La version filté avec un noyau de moyenne.

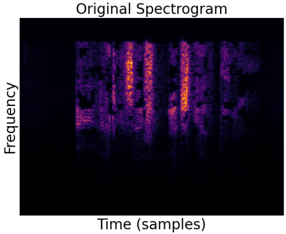


More on filtering

- Low-pass filtering (cut-off at 1kHz) `Listen`.

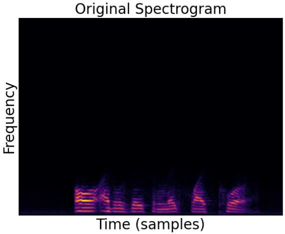


- High-pass filtering (cut-off at 4kHz) `Listen`.

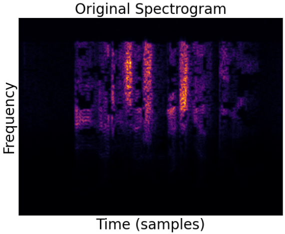


More on filtering

- Low-pass filtering (cut-off at 1kHz) `Listen`.



- High-pass filtering (cut-off at 4kHz) `Listen`.



- There are also band-stop, band-pass filtering..
  - Il existe aussi band-stop, band-pass filtering.

Fast Convolution

- Convolution is an $\mathcal{O}(N^2)$ operation.
- However we can use FFT to get it down to $\mathcal{O}(N \log N)$.
  - On peut prendre avantage de le FFT pour avoir une complexité de $\mathcal{O}(N \log N)$.
- Convolution in time domain, is multiplication in the Fourier Domain.
  - Convolution dans le domaine de temps est multiplication dans le domaine de Fourier.

$$
\begin{aligned}
F(x * w) =& Fx \odot Fw \\
\rightarrow x * w =& F^{-1}(Fx \odot Fw)
\end{aligned}
$$

- Use FFT whenever you can!

Convolution as a Matrix Multiplication

- Convolution can be expressed as a matrix multiplication.
  - Convolution peut-etre exprimée comme une multiplication de matrices.

$$w * x = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} w_1 & 0 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ 0 & w_2 & w_1 & 0 \\ 0 & 0 & w_2 & w_1 \\ 0 & 0 & 0 & w_2 \end{bmatrix}}_{:=C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- C is a Circulant / Toeplitz matrix.

Convolution as a Matrix Multiplication

- Convolution can be expressed as a matrix multiplication.
  - Convolution peut-etre exprimée comme une multiplication de matrices.

$$w * x = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} w_1 & 0 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ 0 & w_2 & w_1 & 0 \\ 0 & 0 & w_2 & w_1 \\ 0 & 0 & 0 & w_2 \end{bmatrix}}_{:=C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- C is a Circulant / Toeplitz matrix.
- And the eigenvectors are sinusoids.

Convolution as a Matrix Multiplication

- Convolution can be expressed as a matrix multiplication.
  - Convolution peut-etre exprimée comme une multiplication de matrices.

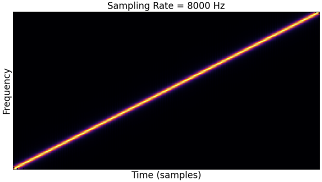
$$w * x = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} w_1 & 0 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ 0 & w_2 & w_1 & 0 \\ 0 & 0 & w_2 & w_1 \\ 0 & 0 & 0 & w_2 \end{bmatrix}}_{:=C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

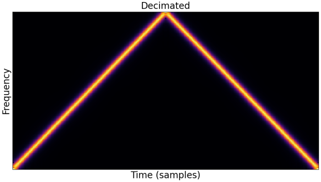
- C is a Circulant / Toeplitz matrix.
- And the eigenvectors are sinusoids.
- Not only that, the eigenvectors form the DFT matrix! `https://web.mit.edu/18.06/www/Spring17/Circulant-Matrices.pdf`

Table of Contents

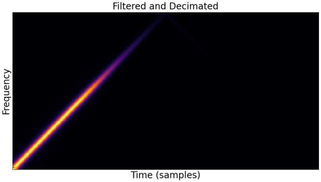
Downsampling

- Let's say we want to downsample. Simple decimation introduces aliasing.
- Si on sous-echantillonne, simple decimation introduit de l'aliasing.





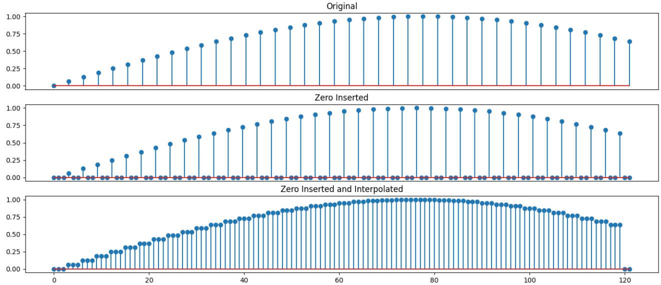
- The solution is to first filter and then downsample.
  - La solution est de filtrer et puis downsampler



- We lose the high-freqs, but we at least avoid aliasing.
  - On perd les hautes frequences mais on moins pas d'aliasing.

Upsampling

- To upsample by a factor of L, the procedure is to first insert $L-1$ zeros, and then to interpolate.
  - Pour upsampler de la facteur L, le procedure est de d'abord inserer $L-1$ zéros, et puis faire de l'interpolation.



Recap

- Signal Representations
  - Time, Frequency, Time-Frequency
- Discrete Fourier Transform
- Short-Time Fourier Transform
- Sampling, Resampling
- Convolution

Furher Reading

- http://www.dspguide.com/pdfbook.htm – nice free book, check it out.
- https://dspguru.com/dsp/howtos/

Next week

- We get started with machine learning (sometimes for signal processing)