

IFT 4031/7031,
Machine Learning for Signal Processing
**Week1: Class Intro,
Linear Algebra Refresher**

Cem Subakan



UNIVERSITÉ
LAVAL



Mila

What is this class?

- What do you think this class is?

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- Is it a Machine Learning class?
- Is it a Signal Processing class?

What is this class?

- What do you think this class is?
- Is it a Machine Learning class?
- Is it a Signal Processing class?
- What is Machine Learning?
- What is Signal Processing?

Signal Processing

■ Here's the wikipedia definition:

Signal processing is an [electrical engineering](#) subfield that focuses on analyzing, modifying and synthesizing *signals*, such as [sound](#), [images](#), [potential fields](#), [seismic signals](#), [altimetry processing](#), and [scientific measurements](#).^[1] Signal processing techniques are used to optimize transmissions, [digital storage efficiency](#), correcting distorted signals, [subjective video quality](#) and to also detect or pinpoint components of interest in a measured signal.^[2]

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- Hm, this kinda sounds like machine learning.

How are signals different than data?



- So, signals are just data?
- Yeah-(ish).

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- Yeah-(ish).
- Why are we calling them signals then?

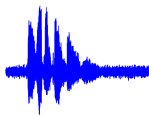
How are signals different than data?



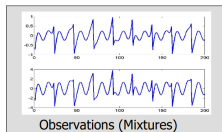
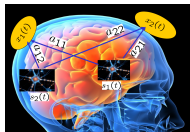
- So, signals are just data?
- Yeah-(ish).
- Why are we calling them signals then?
- When we speak of signals, we refer more to structured data. (Order matters)
- And, saying 'signals', 'signal processing' implies a more Electrical Engineering way to the approach.

Example Signals

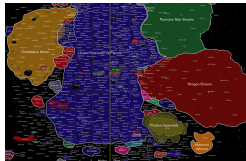
■ Images, Audio/Speech



■ Brains

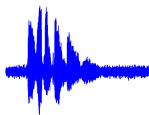


■ Financial Time Series, Graphs

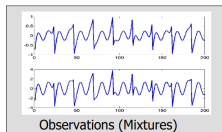
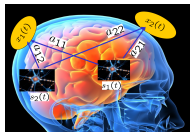


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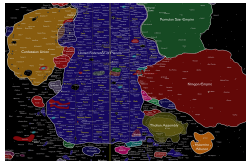
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■ Financial Time Series, Graphs



■ More?

But why bother? Isn't ML what's hip now?

- Yes, ML is extremely popular, and we should embrace that.

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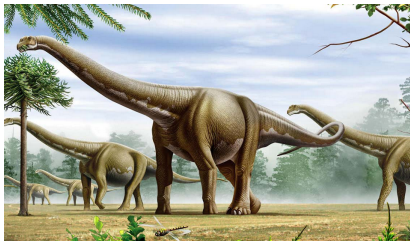
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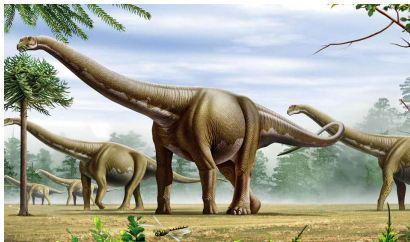
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 - ▶ No!



- ▶ Traditional SP is typically **NOT** statistical, doesn't handle the statistical patterns of the signal well.
- ▶ Traditional SP: Filtering, acquisition, analog-digital-analog conversion, transmission
- ▶ There is statistical signal processing also, but it doesn't go much beyond adaptive filtering.

MLSP: Machine Learning for Signal Processing

- How to build systems that would work with sequences and solve machine intelligence tasks on them?
 - ▶ Various tasks with Speech and Audio: ASR, Speech Enhancement, Music Transcription...

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 - ▶ Understanding Biomedical Sequences

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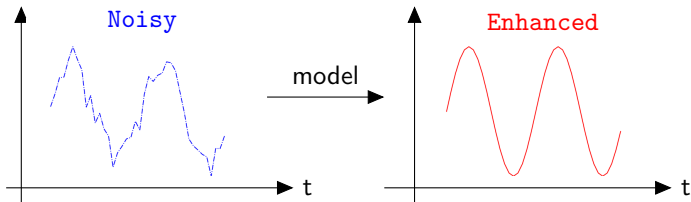
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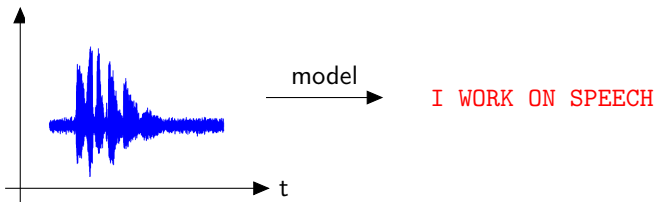
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 - ▶ Generating Videos
 - ▶ More...

Speech and Audio Modeling

■ Speech Enhancement

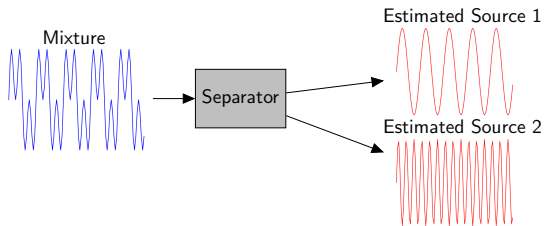


■ Speech Recognition

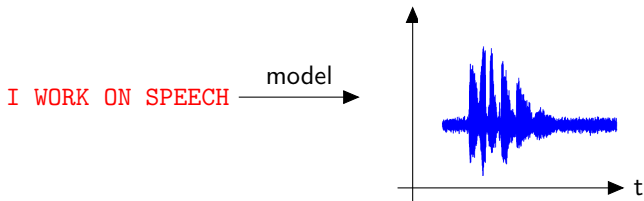


Speech and Audio Modeling

■ Speech Separation

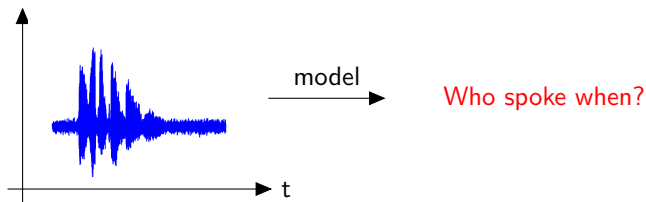


■ Text-to-Speech

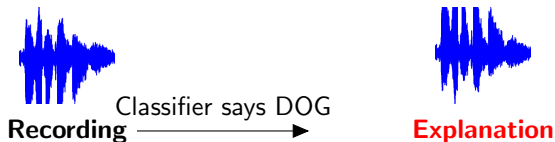


Speech and Audio Modeling

■ Speaker Diarization



■ Neural Network Explanation



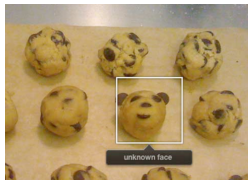
- Other problems: Generating Deep fakes, Detecting deep fakes, Music Source Separation, Music Transcription, Sound Event Detection/Classification...

Speech and Audio Modeling

- Field with huge economic value & job opportunities,
 - ▶ Speech Recognition (e.g. Siri)
 - ▶ Speech Enhancement (e.g. Google meet, Zoom)
 - ▶ Text-to-Speech
 - ▶ Speaker Verification, Spoof Detection(Banks)
 - ▶ Speaker Diarization for Meeting Analysis (Nuance, Microsoft)
 - ▶ Source Separation (e.g. Beatles Rock Band, Meeting Analysis)

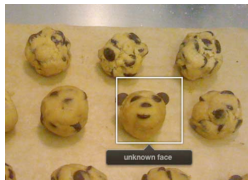
Other real-life applications

- Face recognition



Other real-life applications

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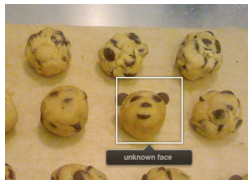


■ Brain-machine interfaces



Other real-life applications

- Face recognition



- Brain-machine interfaces



- Real time bio-signal analysis, learning generative models for bio/medical signals, condition monitoring (mining machines, production machines), Stock market, many more..

About this class

- This is class heavy on practice. How do we make things that work?
- We do not do deep theory in this class.
 - ▶ We will not prove things.
 - ▶ We will not stay Keras level either.
 - ▶ Our goal is to give useful insights, be useful.
- We go fast, our typical lecture could be a class.

Syllabus: Basics

■ Linear Algebra

- ▶ This class

■ Probability

- ▶ Probability Calculus, Random Variables, Bayesian vs Frequentist Principles

■ Signal Processing

- ▶ Signal Representations, Fourier Transform, Sampling

Syllabus: Machine Learning

■ Decompositions

- ▶ PCA, NMF, Linear Regression, Tensor Decompositions

■ Classification

- ▶ Logistic Regression, Maximum Margin, Kernels, Boosting

■ Deep Learning

- ▶ Deep Learning Essentials, Pytorch

■ Optimization

- ▶ Convex optimization
- ▶ Gradient Descent and friends
- ▶ Non-Convex optimization

■ Clustering

- ▶ Kmeans, Spectral Clustering, DBScan

■ Unsupervised Non-linear learning

- ▶ Manifold Learning, Deep Generative Models

■ Time Series Models

- ▶ HMMs, Kalman Filters

Syllabus: Fun Stuff

- Speech Recognition
- Speech Enhancement/Separation
- Text-to-speech
- Representation Learning Methods for Sequences
- Generative Models for Sequences
- Text prompted models (text prompted image / sound generation)
- Neural Network Interpretation Methods
- Graph Signal Processing / ML

Evaluation

■ Homeworks (45%)

- ▶ 3 homeworks, you need to work on these alone!
- ▶ I would like you to typeset math in $\text{\LaTeX}$. So if you don't know it, start learning it!
- ▶ Do not use Generative AI, if you want to learn!
- ▶ You will need to code. But we will reward good quality presentation of results.

■ Weekly Labs (10%)

- ▶ You will work on hands-on application of the things we talk about. TAs will lead the online sessions.
- ▶ Sessions Fridays 12h30 - 14h20. First hour(ish) will be dedicated to the lab. Then it will turn into an office hour with the TAs.

■ Final Project (45%)

Final project

- This will be a mini-conference.
- Each paper will receive 2/3 peer-reviews (from you). We will evaluate the quality of your reviews (5% of your 45% project grade).
- You will work in teams of 2-3 (no more, no less)
- We will ask who did what in the project. So no freeriding!
- Start making friends!
- Mid-October, proposals are due
- Last 1-2 weeks, paper deadline.

Final project

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- Start making friends!
- Mid-October, proposals are due
- Last 1-2 weeks, paper deadline.
 - ▶ We will accept all the papers, and you will make a presentation.
 - ▶ However, you need to do a good job to get a good grade.
 - ▶ If it's a good paper, we can also work together to submit it to a real conference! We can work together towards that.

Communications

- We will have teams page where will have a forum, and you will submit your assignments.
- Be active on the forum, ask questions. Find friends for the project.
- We will do the announcements on teams, so sign-up for it!
- Check <https://ycemsubakan.github.io/mlsp2025.html> for class material.

Instructor: Who am I?

- **Instructor:** Cem Subakan
 - ▶ cem.subakan@ift.ulaval.ca
 - ▶ Assistant Prof. in Computer Science, Mila Associate Academic Member.
 - ▶ Associate member of IEEE MLSP Technical Committee. Was a general Chair of MLSP 2025 conference.
 - ▶ Just send me a message you if you want to meet.
- I work on machine learning for Speech and Audio.
 - ▶ Interpretability, Explainability
 - ▶ Speech Separation & Enhancement
 - ▶ Multi-Modal Learning
 - ▶ Continual Learning
 - ▶ Probabilistic Machine/Deep Learning
- I review for many major conferences, involved in the organization of several workshops.
- I have written a lot of papers involving MLSP topics, worked with many people, also saw the industry side of things.

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- Office hours: By appointment. DM me on teams to take an appointment.

You can submit your work to MLSP 2025!



You can submit your work to MLSP 2025!



You can submit your work to MLSP 2025!



There are other conferences now we can submit!

Who are the TAs?

- Sara Karami
 - ▶ sara.karami.1@ulaval.ca
- Mathieu Bazinet
 - ▶ mathieu.bazinet.2@ulaval.ca
- Benjamin Léger
 - ▶ benjamin.leger.1@ulaval.ca

Who are the TAs?

- Sara Karami
 - ▶ sara.karami.1@ulaval.ca
- Mathieu Bazinet
 - ▶ mathieu.bazinet.2@ulaval.ca
- Benjamin Léger
 - ▶ benjamin.leger.1@ulaval.ca
- TAs will hold the online lab sessions (Fridays 12h30-14h20)
- Advice:
 - ▶ If you need help do not bombard them at the last minute. Seek help early.

Who are you?

■ Name, department, grad/undergrad?

- ▶ What are your interests?
- ▶ Hint: Take notes, and contact the person if something picks your interest.

Table of Contents

Linear Algebra Refresher

- Basics

- Array Manipulation

- More linear algebraic concepts

- Decompositions

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- Vector, x , of length L
$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$

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- Matrix, x of size $L \times M$

$$x = \begin{bmatrix} x_{1,1} & \dots & x_{1,M} \\ \vdots & \vdots & \vdots \\ x_{L,1} & \dots & x_{L,M} \end{bmatrix} \\ = \begin{bmatrix} x_1 & \dots & x_M \end{bmatrix}$$

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- Tensor, x of size $L \times M \times N$

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Scalars, Vectors, Matrices, Tensors

- Scalar, x , just a number.
0th order tensor.

- Vector, x , of length L
$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$
1th order tensor.

- Matrix, x of size $L \times M$
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$$= \begin{bmatrix} x_1 & \dots & x_M \end{bmatrix}$$
2nd order tensor.

- Tensor, x of size $L \times M \times N$

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3rd order tensor.

How do we represent signals as these?

■ Sounds, Time Series

$$x^T = [x_1 \quad \dots \quad x_L] = \left[\text{[audio waveform]} \right]$$

■ Images

$$X = \begin{bmatrix} x_{1,1} & \dots & x_{1,M} \\ \vdots & \vdots & \vdots \\ x_{L,1} & \dots & x_{L,M} \end{bmatrix} = \left[\text{[image of a man in a Star Trek uniform]} \right]$$

■ Videos as tensors.. and so on..

Table of Contents

Linear Algebra Refresher

Basics

Array Manipulation

More linear algebraic concepts

Decompositions

Index/Array Notation

- We need good ways to communicate operations on these objects.
- **Option 1:** Index Notation
 - ▶ Micro-level and detailed, but not very compact
- **Option 2:** Array Notation
 - ▶ Compact but abstracts away the details

Index Notation

- We define the elements in index form.
 - ▶ Element-wise multiplication:

$$c_i = a_i b_i$$

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$$C_{ik} = \sum_j A_{ij} B_{jk}$$

- ▶ Some random tensor operations

$$C_{im} = \sum_{j,l,k} A_{ijlk} B_{mjlk}, \quad c = \sum_{i,j} A_{ij} B_{ij}$$

Array Notation

- We define the elements in index form.

- ▶ Element-wise multiplication:

$$c = a \odot b, c \in \mathbb{R}^L$$

- ▶ Inner product of vectors

$$c = \langle a, b \rangle = a^\top b, c \in \mathbb{R}$$

- ▶ Outer product of vectors

$$c = a \otimes b = ab^\top, c \in \mathbb{R}^{L \times M}$$

- ▶ Matrix-vector product

$$c = Ab, c \in \mathbb{R}^L$$

- ▶ Matrix multiplication

$$C = AB, C \in \mathbb{R}^{L \times M}$$

- ▶ Some random tensor operations

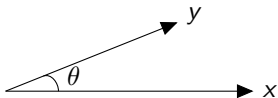
$$C = A \times_{jlk} B, C \in \mathbb{R}^{L \times M} \quad c = A \times_{i,j} B, c \in \mathbb{R}$$

Index vs Array Notation

- Index Notation is very specific, not ambiguous
- But the array notation makes it possible to manipulate the operations with ease. (E.g. gradient calculations)

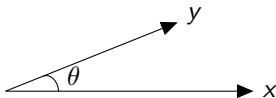
The dot product

■ $c = \sum_i a_i b_i = a^\top b = \|a\| \|b\| \cos \theta$



The dot product

■ $c = \sum_i a_i b_i = a^\top b = \|a\| \|b\| \cos \theta$



■ Note that,

$$\theta = \arccos \left(\frac{a^\top b}{\|a\| \|b\|} \right)$$

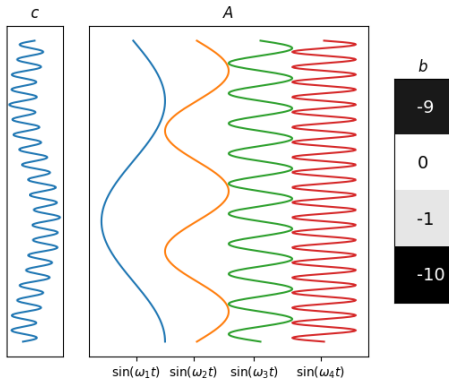
■ So, dot product is a great tool to measure similarity.

Matrix-Vector Product

- $c = Ab$, or $c_i = \langle A_{i,:}, c \rangle = \sum_j A_{ij}c_j$. A is a matrix, b is vector. c is a what?

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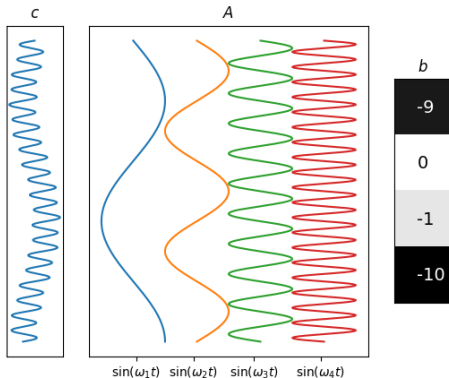


Matrix-Vector Product - 2nd interpretation

- It's a series of dot products.
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Matrix-Matrix Product

- It's a series of Matrix-vector products. (or series of inner products on a grid)
- $C = AB$, or $C_{ij} = \sum_k A_{ik} B_{kj}$, or $C_{ij} = A_{i,:}^\top B_{:,j}$

■

$$C = \begin{bmatrix} A_{1,:}^\top \\ A_{2,:}^\top \\ A_{3,:}^\top \end{bmatrix} \begin{bmatrix} B_{:,1} & B_{:,2} & B_{:,3} \end{bmatrix} = \begin{bmatrix} A_{1,:}^\top B_{:,1} & A_{1,:}^\top B_{:,2} & A_{1,:}^\top B_{:,3} \\ A_{2,:}^\top B_{:,1} & A_{2,:}^\top B_{:,2} & A_{2,:}^\top B_{:,3} \\ A_{3,:}^\top B_{:,1} & A_{3,:}^\top B_{:,2} & A_{3,:}^\top B_{:,3} \end{bmatrix}$$

Matrix-Matrix Product

- It's a series of Matrix-vector products. (or series of inner products on a grid)

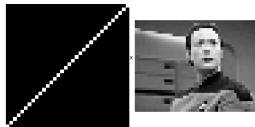
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- Not any pair of two matrices can be multiplied. You need to have equal number of columns from A , number rows from B .
- Master this, it will help! This has to become muscle memory.

Visualize the matrix product



Visualize the matrix product



Visualize the matrix product



Visualize the matrix product



Visualize the matrix product



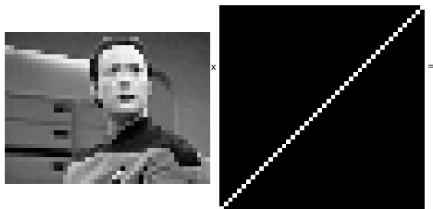
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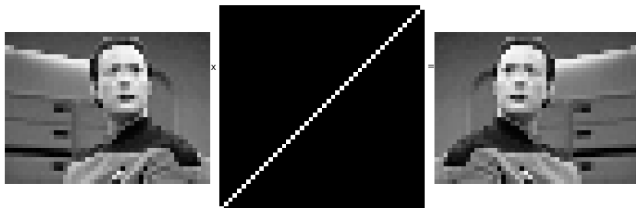
Visualize the matrix product



Multiplying from the other side



Multiplying from the other side



Reversing on the horizontal axis

Einstein Notation

- Let's go beyond matrices!
- $C_{i,j} = \sum_{l,k} A_{i,l,k} B_{l,j,k}$

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$$A_{i,l,k}, w_{l,j,k} \rightarrow C_{i,j}$$

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- $A_{i,l}, B_{l,j} \rightarrow C_{i,j}$

Let's do more Einstein stuff

- Element-wise multiplication:

$$c = a \odot b, c \in \mathbb{R}^L$$

- Inner product of vectors

$$c = \langle a, b \rangle = a^\top b, c \in \mathbb{R}$$

- Outer product of vectors

$$c = a \otimes b = ab^\top, c \in \mathbb{R}^{L \times M}$$

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- Some random tensor operation

$$C = A \times_{jlk} B, \quad C \in \mathbb{R}^{L \times M} \quad c = A \times_{i,j} B$$

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$$C = A \times_{jlk} B, \quad C \in \mathbb{R}^{L \times M} \quad c = A \times_{i,j} B$$

$$A_{ijkl}, B_{mjlk} \rightarrow C_{im}$$

Implementing Einstein products is easy in Python

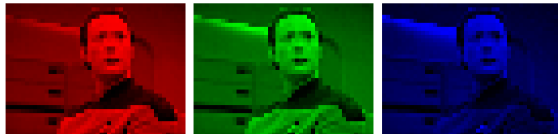
■ Batch Matrix Multiplication

$$A_{bij}B_{bjk} \rightarrow C_{bik}$$

```
C = torch.einsum('bij ,bjk->bik', A, B)
```

Application of Tensor Operations

- RGB images



- Let us apply a matrix multiplication to each channel, and then average over the channels.

Application of Tensor Operations

- In Index Notation

$$C_{ij} = \sum_{k,c} \underbrace{B_{ik}}_{\text{Matrix}} \underbrace{A_{kjc}}_{\text{image}} \underbrace{w_c}_{\text{WtOverCh.}}$$

- Notice that this notation can handle multilinear operations.

Application of Tensor Operations

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- In Einstein Notation:

$$B_{ik}, A_{kjc}, w_c \rightarrow C_{ij}$$

- Notice that this notation can handle multilinear operations.

Application of Tensor Operations

■ First step

$$B_{ik}, A_{kjc} \rightarrow T_{ijc}$$



■ Second step

$$T_{ijc} w_c \rightarrow C_{ij}$$



Let's also see some reshaping operations

■ Vectorization:

$$\text{vec} \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

■ The 'Diag' Operation:

$$\text{Diag} \left(\begin{bmatrix} a_1 & a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

■ The 'Reshape' Operation:

$$\text{Reshape}_{32} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{22} \\ a_{21} & a_{13} \\ a_{12} & a_{23} \end{bmatrix}$$

Kronecker Product

- It's sort of an outer product but has a specific shape,

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- Let's visualize this,

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \otimes \text{Image} = \begin{bmatrix} \text{Image} & 4 \times \text{Image} \\ 0 & 2 \times \text{Image} \end{bmatrix}$$


Why Bother?

- Sometimes matrix algebra is compact and powerful.

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- The matrix form could be helpful when calculating gradients, and coming up with efficient implementations.
- Einsum is not as optimized as matrix multiplication.

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Linear Algebra Refresher

Basics

Array Manipulation

More linear algebraic concepts

Decompositions

Matrix inverse

- Let's think about a linear system,

$$Ax = b$$

$$\rightarrow A^{-1}Ax = x = A^{-1}b$$

- Is A^{-1} always defined?

Matrix inverse

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$$Ax = b$$

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- Is A^{-1} always defined?
- First, A needs to be square.
- Second, it needs to be full rank. Columns of A need to be linearly independent.

Matrix pseudoinverse

- Let's have the same linear system, but with a rectangular A matrix,

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Matrix pseudoinverse

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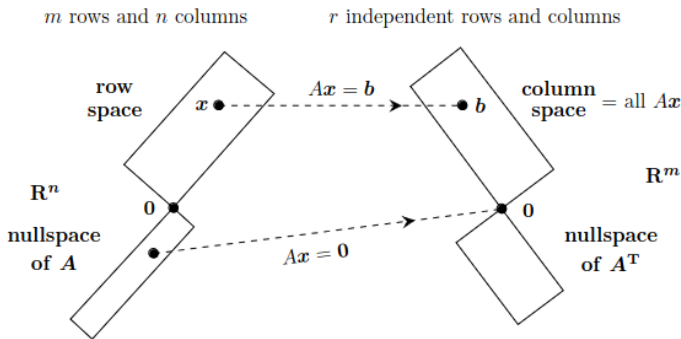
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- $A^\dagger := (A^\top A)^{-1} A^\top$. This is known as the pseudo inverse.
- This is essentially least squares. (We will show that later)

Four Fundamental Subspaces in Linear Algebra



BIG PICTURE OF LINEAR ALGEBRA

row space $\perp$ nullspace

column space of $A \perp$ nullspace of A^T

row rank = column rank = r

Image Taken from Gilbert Strang's 'Introduction to Linear Algebra' book.

- l_2 norm: $\|x\|_2 = \sqrt{\sum_j x_j^2}$. Also known as Euclidean Norm.
- l_1 norm: $\|x\|_1 = \sum_j |x_j|$.
- l_p norm: $\|x\|_p = \sqrt[p]{\sum_j |x_j|^p}$.
- $\text{tr}(A) = \sum_i A_{ii}$, it's basically the sum of diagonal elements.
Do not underestimate this.

Norms, trace

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- Frobenius norm: $\|X\|_F = \sqrt{\sum_i \sum_j |X_{ij}|^2} = \sqrt{\text{tr}(XX^\top)}$

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- Index notation helps to derive these. Otherwise you can just pattern match from the matrix cookbook.
- We are just giving an idea here with simple examples. We will see these more in real action later. (hint: backprop)

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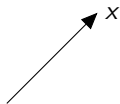
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More linear algebraic concepts

Decompositions

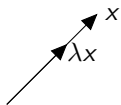
Eigenvalues / Eigenvectors

■ $Ax = \lambda x$



Eigenvalues / Eigenvectors

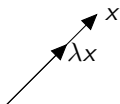
■ $Ax = \lambda x$



■ Note that x doesn't change its direction.

Eigenvalues / Eigenvectors

- $Ax = \lambda x$



- Note that x doesn't change its direction.
- Eigenvectors are 'characteristic' directions for the system described by A .

Finding the Eigenvectors

- The 'Linear Algebra Class Way':
- Let's have this matrix

$$A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$$

- Calculate the determinant (why?)

$$\det(A - \lambda I) = \begin{vmatrix} 0.8 - \lambda & 0.4 \\ 0.2 & 0.6 - \lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.40$$

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- Then we find vectors in the null space of $A - \lambda I$
- $A - I = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix} v = 0$, find a non-zero vector v such that the equation is satisfied. $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

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- Here are the power iterations starting from $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



- To get all the eigenvectors we can deflate the matrix. Just subtract v , and repeat the process..

Ok, but how is this a decomposition?

- $AV = V\Lambda$, where columns of V are the eigenvectors, and Λ is a diagonal matrix with eigenvalues on the diagonal.

Ok, but how is this a decomposition?

- $AV = V\Lambda$, where columns of V are the eigenvectors, and Λ is a diagonal matrix with eigenvalues on the diagonal.
- And here's the decomposition $A = V\Lambda V^{-1}$.
- But notice that this decomposition is only defined for square matrices.

Singular Value Decomposition

- Let us given a matrix of size X in $\mathbb{R}^{M \times N}$.
- $X = U\Sigma V^T$, $U \in \mathbb{R}^{M \times M}$ and is orthogonal $U^T U = I$, $\Sigma \in \mathbb{R}^{M \times N}$ is a matrix with non-zero elements on the main diagonal, and $V \in \mathbb{R}^{N \times N}$, and is orthonal $VV^T = I$.

$$\boxed{X_{M \times N}} = \boxed{U_{M \times M}} \boxed{\Sigma_{M \times N}} \boxed{V_{N \times N}^T}$$

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- An alternative way of viewing it is $X = \sum_{k=1}^M \sigma_k u_k v_k^T$. Note that we can cut the sum short, and keep the biggest singular values! (set $X = \sum_{k=1}^K \sigma_k u_k v_k^T$, $K \leq M$)

$$\boxed{X_{M \times N}} = \boxed{U} \boxed{\Sigma} \boxed{V^T}$$

Relationship between SVD and Eigenvalue decomposition

- $X = U\Sigma V^T$, this is SVD.

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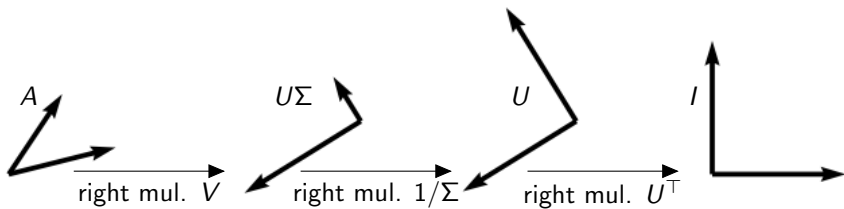
Relationship between SVD and Eigenvalue decomposition

- $X = U\Sigma V^T$, this is SVD.
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Relationship between SVD and Eigenvalue decomposition

- $X = U\Sigma V^\top$, this is SVD.
- $XX^\top = U\Sigma \underbrace{V^\top V}_I \Sigma^\top U^\top = U\Sigma^2 U^\top$.
- Singular vectors U of X , are the eigenvectors of $XX^\top$.
- Singular values σ_k of X , are the square root of eigenvalues of $XX^\top$.
- For positive semi-definite matrices, SVD and eigenvalue decomposition are equivalent.

Geometric Interpretation of SVD



List of Decompositions

- **LU decomposition:** $X = LU$, L is lower triangular, U is upper triangular.
- **QR decomposition:** $X = QR$, Q is a matrix with orthonormal columns, R is an upper triangular matrix.
- **Eigenvalue decomposition:** $X = U\Lambda U^{-1}$, columns of U are eigenvectors of X , which is square (diagonalizable) matrix.
- **Singular value decomposition:** $X = U\Sigma V^T$, columns of U , and V have orthonormal columns. Defined for any matrix.

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- **Singular value decomposition:** $X = U\Sigma V^T$, columns of U , and V have orthonormal columns. Defined for any matrix.
- There's more, e.g. Cholesky, NMF, CR, ICA, ...

List of special type of matrices we'll see in this class

- **Rotation matrices**
- **Markov matrices** (Probability Transition Matrices)
- **Transform matrices** (Fourier Transform, Convolution,...)
- **Covariance matrices** (Define a Multivariable Random Variable)
- **Adjacency matrices** (Define a Graph)

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- We saw how data/signals can be represented.



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- We saw how the data can be manipulated. (Vector, Matrix, Tensor Operations)



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- We saw how data/signals can be represented.



- We saw how the data can be manipulated. (Vector, Matrix, Tensor Operations)



- We took a glimpse into how we can decompose signals.
- We gave a crude summary into what we need from Linear Algebra.

Recommended Reading

- Gilbert Strang, Introduction to Linear Algebra,
https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/video_galleries/video-lectures/,
<https://math.mit.edu/~gs/linearalgebra/ila5/indexila5.html>
- Trefethen and Bau, Numerical Linear Algebra,
<https://people.maths.ox.ac.uk/trefethen/text.html>
- Matrix Cookbook,
<http://www2.imm.dtu.dk/pubdb/edoc/imm3274.pdf>

What's Next

- Probability Calculus, Random Variables, Multi-dimensional Distributions
- Exponential Family Distributions
- Maximum Likelihood, MAP, Bayesian parameter estimation principles
- Labs are starting next week! (first one is Sept. 12)