IFT 4030/7030, Machine Learning for Signal Processing

Week 9: Time Series Modeling

Cem Subakan

Admin

- Homework 2 is out. You can do it all on the notebook.
  - Devoir 2 est publié. Vous pouvez tout faire sur le notebook.
- If you have doubts on your projects contact me. Thanks for those of you who already did. If you do a good project it will help you in your job search / grad school search.
  - Si vous avez des doutes sur vos projets, c'est très important que vous me contactez. Merci pour ceux qui a déjà fait. Si vous faites un bon projet ça va vous aider à trouver un job, ou un grad student position...

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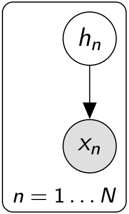
- I will share a sign-up sheet soon for project presentations.
  - Je partager un sign-up sheet bientot pour les presentations.
- This week: / Cette semaine: Time series!

Our view of data so far (well, mostly)

- IID: (not the institute) Independent and Identically Distributed
- Remember last week? / Souvenez-vous de ça de la semaine dernière?



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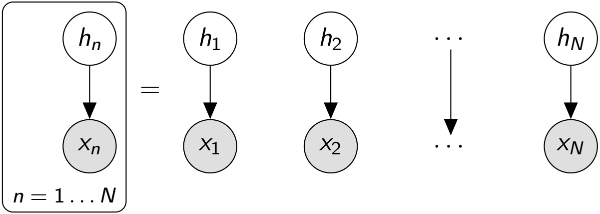

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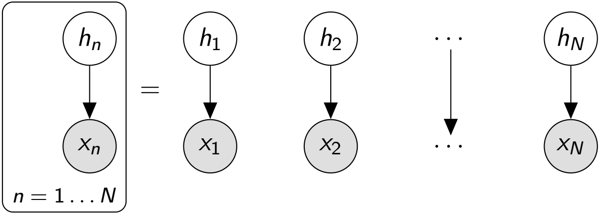
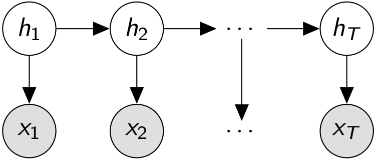
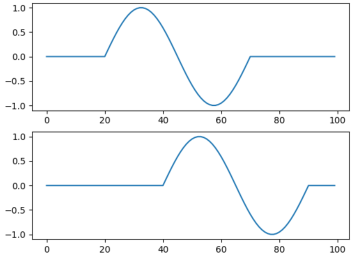


- Well, this week we will have this instead:
  - Bon, cette semain on aura ça:



Motivation

- Are these signals similar? / Ces deux signals, sont-ils similaires?



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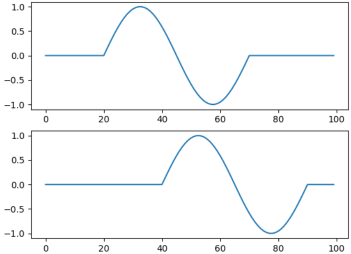

- If we calculate a distance in the 100d space they are distant.
  - Si on calcule une distance dans une espace de 100d, ils sont distants.

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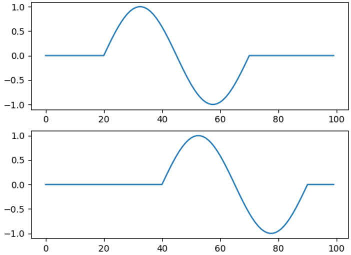
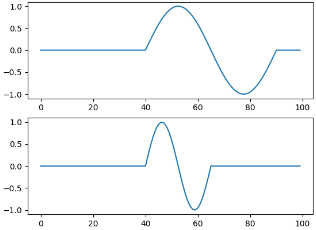


- If we calculate a distance in the 100d space they are distant.
  - Si on calcule une distance dans une espace de 100d, ils sont distants.
- What can we do?
  - Que peut-on faire?

How about these?

- Are these similar? / Sont-ils similaires?



- How about these? / Et ces deux?

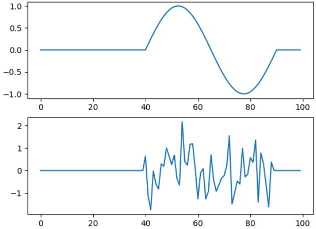


Table of Contents

Dynamic Time Warping

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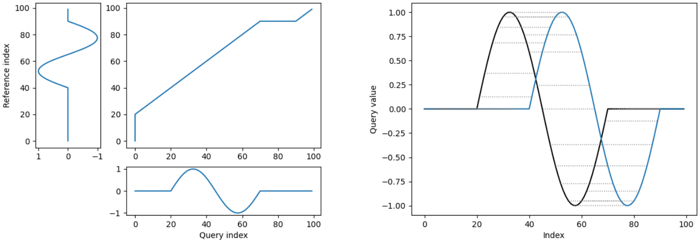
Dynamic Time Warping

- We can to find a 'warping function' between two signals to match them.
  - On peut trouver une fonction de 'warping' entre les deux signaux pour les matcher.



Dynamic Time Warping

- The problem formulation / La formulation du problème

$$\text{DTW}(x, y) = \min_{\pi} \sqrt{\sum_{(i,j)\in\pi} d(x_i, y_i)}$$

  - where, π is a path defined over the indices. / où π est un path défini sur les indices.
- More formally/Plus formellement $\pi = [(i_0, j_0), (i_1, j_1), \ldots, (i_K, j_K)]$.
- Constraints: / Contraints:

$$\begin{aligned} \pi_0 &= (0, 0) \\ \pi_K &= (n-1, m-1) \\ i_{k-1} \leq i_k &\leq \ i_{k-1} + 1 \\ j_{k-1} \leq j_k &\leq \ j_{k-1} + 1 \end{aligned}$$

The dynamic programming solution

- The recursion / La recursion

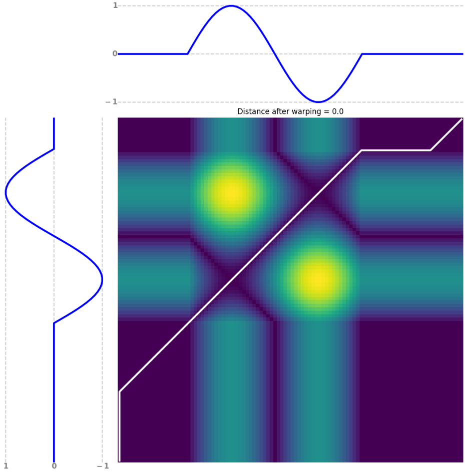
$$C_{i,j} = d(x_i, y_j) + \min(C_{i-1,j}, C_{i,j-1}, C_{i-1,j-1})$$

The dynamic programming solution

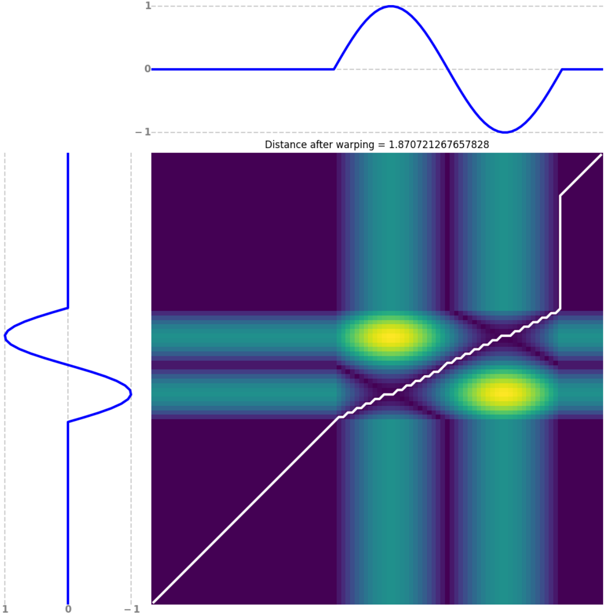
- The recursion / La recursion

$$C_{i,j} = d(x_i, y_j) + \min(C_{i-1,j}, C_{i,j-1}, C_{i-1,j-1})$$

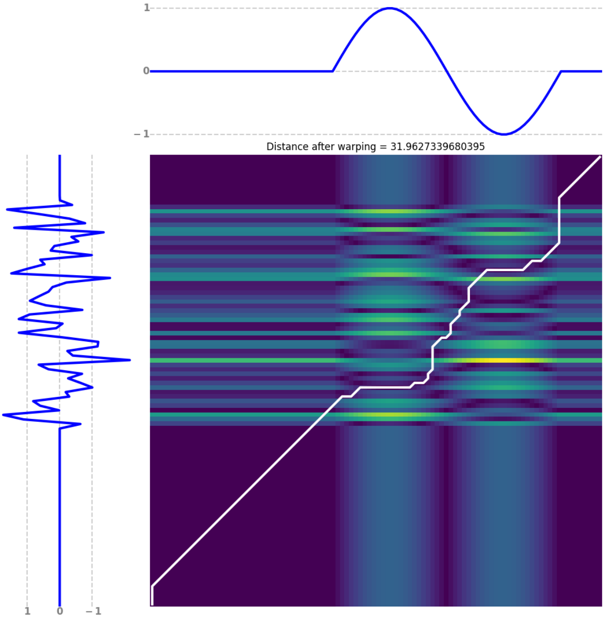
- Optimal path after backtracking / La trace optimale après backtracking



DTW examples

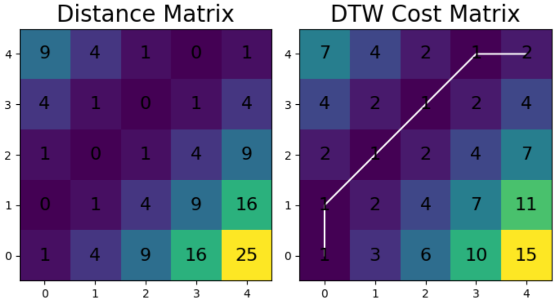


DTW examples



DTW on two arrays

- $x = [0, 1, 2, 3, 4]$, $y = [1, 2, 3, 4, 5]$.



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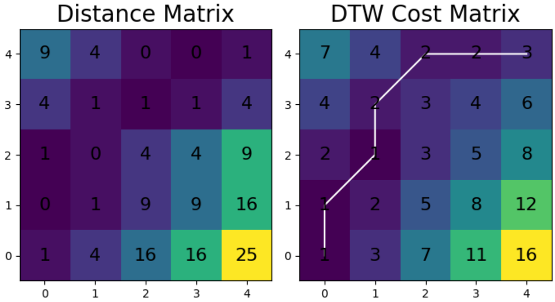
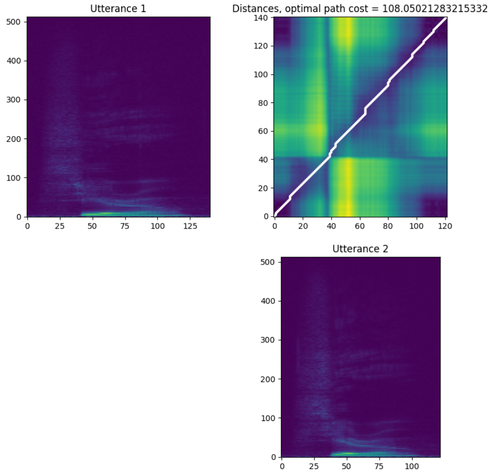


DTW on Audio

- DTW on same person saying 'zero' two different times.
  - DTW sur la meme personne qui dit 'zero' deux differentes fois.



DTW on Audio

- DTW on same person saying 'zero' and 'one'
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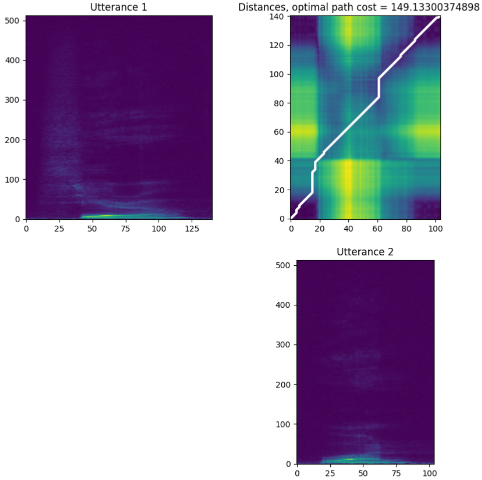
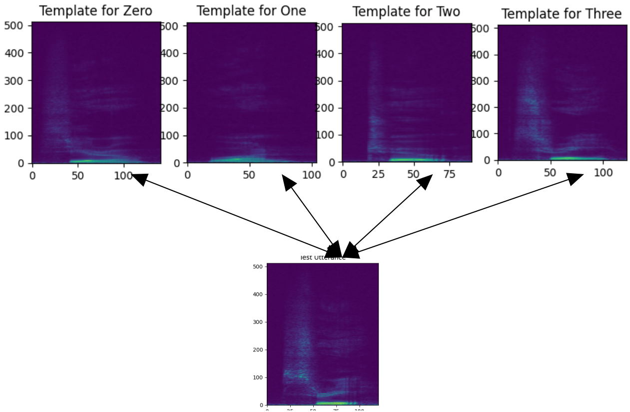


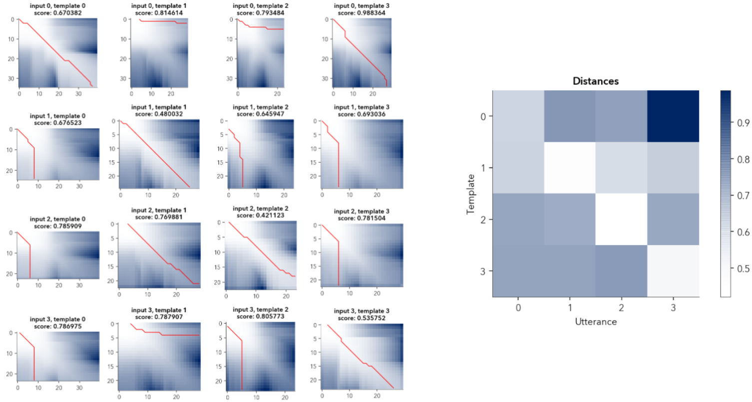
  - Do you notice something? / Vous remarquez quelque chose qui peut etre utile?

Using DTW for sequence classification

- An extremely basic speech classifier / Un classificateur vocale extremement simple.
- We can store templates for each class and then assign to the one with smallest DTW cost.
  - On peut garder des templates pour chaque classe, et assigner le séquence test à la classe avec le cout DTW le plus petit.



DTW classification in action



Taken from UIUC MLSP class slides

To summarize DTW

- It's a distance over time-series, and having that is awesome.
  - C'est une distance sur des time-series, et c'est magnique d'en avoir un.
- You can train a neural net with it, do clustering (maybe? can you? what would be a problem?).
  - Vous pouvez entrainer un réseau neural avec ca, ou clustering (peut-etre vous pouvez?)

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  - Vous pouvez entrainer un réseau neural avec ca, ou clustering (peut-etre vous pouvez?)
- You can even dub movies! / Vous pouvez meme faire du doublage avec DTW.

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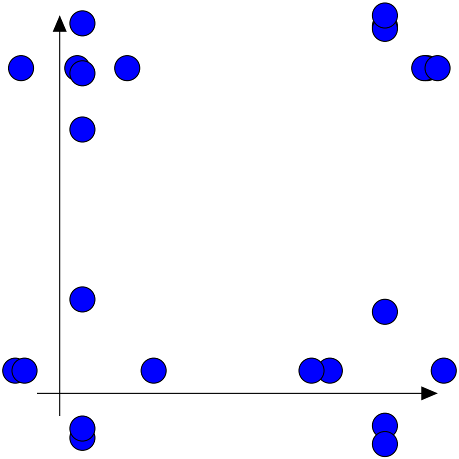
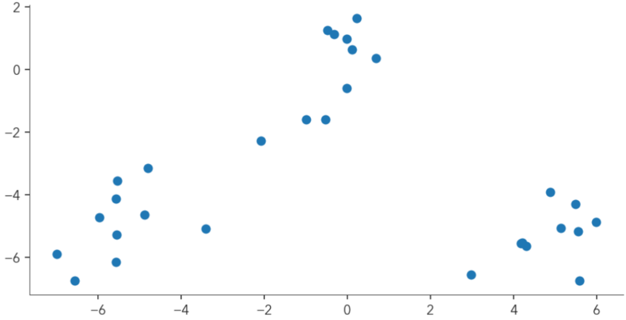
A model over time series

- DTW is nice and all, but it's just a distance, it's not a model.
  - DTW est bon mais ce n'est qu'une distance, n'est pas une modèle.
- Remember the duality between likelihood / and distances (e.g. Gaussian and Euclidean distance)

Let's remember GMMs



GMMs with time



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GMMs with time

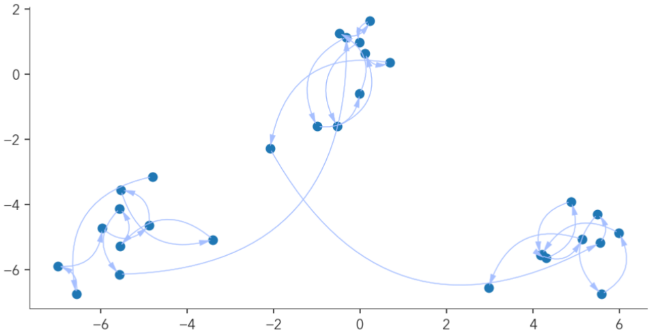
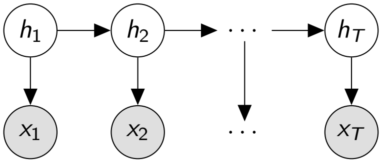


Image taken from UIUC MLSP class

Tired of IID models? HMMs

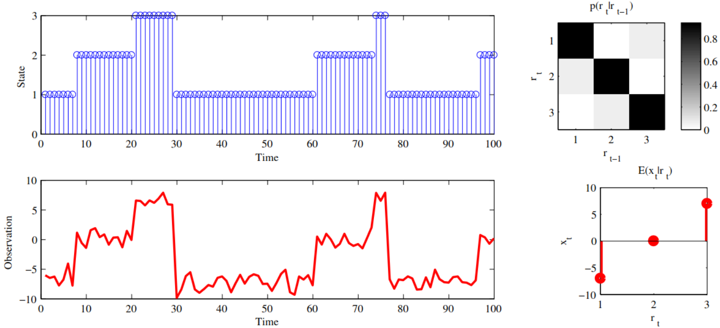
- Model / Modèle:



$$h_n | h_{n-1} \sim \text{Discrete}(A(:, h_{n-1}))$$
$$x_n | h_n \sim p(x_n | h_n, O)$$

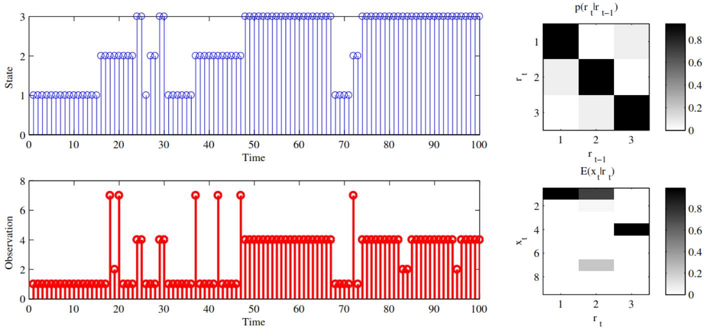
- $h_n \in \{1, \ldots, K\}$, latent variables (embeddings). Difference from before is h_n and h_{n+1} are connected!
  - Différence avec avant est que les variables latent sont connéctés.
- $x_n \in \mathbb{R}^L$, observed data items / les données observées.
- The parameters $\theta = \{O, A\}$ / Les paramètres.
- $O \in \mathbb{R}^{L \times K}$, the emission matrix, $A \in \mathbb{R}^{K \times K}$, the transition matrix.
- Learning is conceptually all the same. Just that E-step is little different.
  - L'apprentissage est meme qu'avant. C'est juste que E-step est un peu différent.

Continuous data generated from an HMM



In this case $p(x_t|r_t, O) = \mathcal{N}(x_n; \mu_{r_t}, \sigma^2 I)$.

Discrete data generated from an HMM



In this case $p(x_t | r_t, O) = \text{Discrete}(x_t; O[:, r_t])$.

Statistical Problems to solve with HMMs

- **Inference / Evaluation**
  - How do we calculate $p(x_{1:T}|\theta)$. / Comment est-ce qu'on calcule ce marginale?
- **Decoding**
  - What are the optimal state values (not probabilities) given $x_{1:T}$ and a learnt model. / Comment obtiens-t-on les valeurs des états optimales?
- **Learning**
  - Given a sequence $x_{1:T}$, how do we learn the optimal model parameters? / Étant donné une séquence $x_{1:T}$ comment est-ce qu'on peut apprendre les paramètres optimales du modèle?

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Inference in HMMs

- The precise inference question / La question précise pour l'inférence.

$$p(x_{1:T}|\theta) = \sum_{h_{1:T}} p(x_{1:T}, h_{1:T}|\theta)$$

Inference in HMMs

- The precise inference question / La question précise pour l'inférence.

$$p(x_{1:T}|\theta) = \sum_{h_{1:T}} p(x_{1:T}, h_{1:T}|\theta)$$
$$= \sum_{h_{1:T}} \prod_{t=1}^{T} p(x_t|h_t)p(h_t|h_{t-1})$$

Inference in HMMs

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p(x_{1:T}|\theta) &= \sum_{h_{1:T}} p(x_{1:T}, h_{1:T}|\theta) \\
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&= \sum_{h_T} \cdots \sum_{h_2} \sum_{h_1} p(x_T|h_T) p(h_T|h_{T-1}) \ldots p(x_2|h_2) p(h_2|h_1) p(h_1)
\end{aligned}
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\end{aligned}
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- This is a huge sum! / C'est une operation immense.

Inference in HMMs

- The precise inference question / La question précise pour l'inférence.

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- This is a huge sum! / C'est une operation immense.
- What can we do? / Qu'est-ce qu'on peut faire?

Inference in HMMs

- The forward inference: (The filtering density)
  - Inférence en avançant

$$\alpha(h_t) := p(x_{1:t}, h_t)$$

- The backward inference:
  - Inférence en réculant

$$\beta(h_t) := p(x_{t+1:T} | h_t)$$

The Dynamic Programming Solution (Again)

$$\alpha(h_t) = p(x_t|h_t) \sum_{h_{t-1}} p(h_t|h_{t-1}) \underbrace{p(x_{t-1}|h_{t-1}) \dots \underbrace{p(x_2|h_2) \sum_{h_1} p(h_2|h_1) p(x_1|h_1) \underbrace{p(h_1)}_{\alpha(h_1)}}_{\alpha(h_2)}}_{\alpha(h_{t-1})}$$

$$\beta(h_t) = \sum_{h_{t+1}} p(h_{t+1}|h_t) p(x_{t+1}|h_{t+1}) \dots \underbrace{\underbrace{\sum_{h_T} p(h_T|h_{T-1}) p(x_T|h_T) \underbrace{1}_{\beta(h_T)}}_{\beta(h_{T-1})}}_{\beta(h_{t+1})}$$

The forward and backward recursions

- The forward recursion / La recurrence en avançant:

$$\alpha(h_t) = p(x_t|h_t) \sum_{h_{t-1}} p(h_t|h_{t-1})\, \alpha(h_{t-1})$$

- The backward recursion / La recurrence en reculant:

$$\beta(h_t) = \sum_{h_{t+1}} p(h_{t+1}|h_t) p(x_{t+1}|h_{t+1})\, \beta(h_{t+1})$$

Ok but what happened to the likelihood question?

- Note:

$$\alpha(h_T) = p(x_1, x_2, \ldots, x_T, h_T)$$

- So how do we get $p(x_{1:T})$?

Ok but what happened to the likelihood question?

- Note:

$$\alpha(h_T) = p(x_1, x_2, \ldots, x_T, h_T)$$

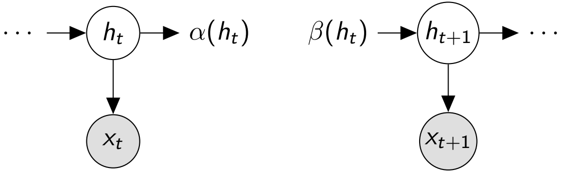
- So how do we get $p(x_{1:T})$?
- $p(x_{1:T}) = \sum_{h_T} p(x_{1:T}, h_T) = \sum_{h_T} \alpha(h_T)$.

Why do we need the backward?

- $\alpha(h_t)$ are "forward messages". $\beta(h_t)$ are "backward messages". One forward pass and one backward pass is sufficient since, / Une passe en avançant et une passe en réculant suffissent parce que,

$$
\begin{aligned}
p(h_t|x_{1:T}) \propto & \, p(h_t, x_{1:T}) \\
= & \, p(h_t, x_{1:t})p(x_{t+1:T}|h_t) \\
= & \, \alpha(h_t)\beta(h_t)
\end{aligned}
$$

- Traditionally (EE traditions), $\alpha_{1:T}$ is known as the filtering density. $\gamma_{1:T} := \alpha_{1:T}.*\beta_{1:T}$ is the smoothing density (la densité smoothing).



Forward Pass in Action

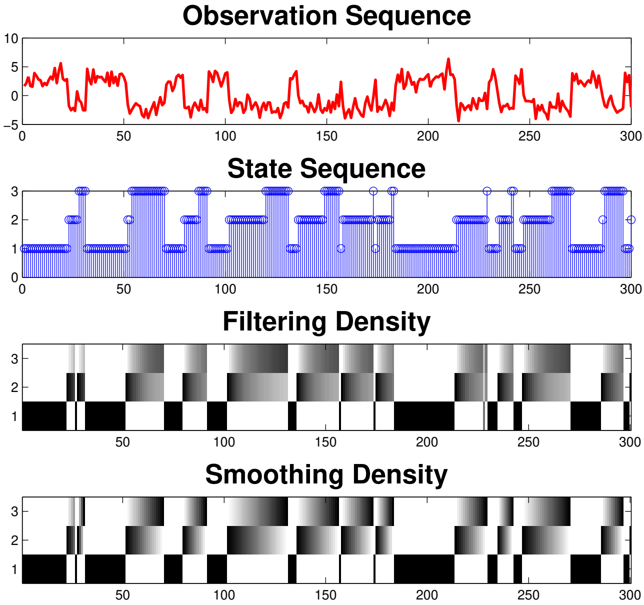


Image taken from Boun CMPE58K slides

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The learning question is the same as GMMs

- The learning question / La problématique d'apprentissage

$$\begin{aligned}\widehat{\theta} &= \arg\max_{\theta} p(x_{1:N}|\theta) \\ &= \arg\max_{\theta} \sum_{h_{1:N}} p(x_{1:N}, h_{1:N}|\theta)\end{aligned}$$

The learning question is the same as GMMs

- The learning question / La problématique d'apprentissage

$$\widehat{\theta} = \arg\max_{\theta} p(x_{1:N}|\theta) = \arg\max_{\theta} \sum_{h_{1:N}} p(x_{1:N}, h_{1:N}|\theta)$$

- Write down/Écrit log-likelihood:

$$\log p(x_{1:N}|\theta) = \log \sum_{h_{1:N}} \frac{p(x_{1:N}, h_{1:N}|\theta)}{q(h_{1:N})} q(h_{1:N}) = \log \mathbb{E}_q \left[\frac{p(x_{1:N}, h_{1:N}|\theta)}{q(h_{1:N})} \right]$$

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The learning question is the same as GMMs

- The learning question / La problématique d'apprentissage

$$\widehat{\theta} = \arg\max_{\theta} p(x_{1:N}|\theta) \\ = \arg\max_{\theta} \sum_{h_{1:N}} p(x_{1:N}, h_{1:N}|\theta)$$

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- Except the fact that the posterior distribution is $q(h_t|x_{1:T}) = p(h_t|x_{1:T})$. Not $p(h_t|x_t)$ (unlike the GMM case. / La différence du cas des GMMs est que maintenant le posterior ne se factorise pas sur temps.

EM Algorithm for HMMs

Randomly initialize θ.
while Not converged **do**
  E-step:
    Do a Forward and backward pass. Get all $\alpha(h_t)$ and $\beta(h_t)$.
  M-step:
    $\widehat{\mu}_k = \frac{\sum_{t=1}^{T} \mathbb{E}_q[h_t=k]x_t}{\sum_{t=1}^{T} \mathbb{E}_q[h_t=k]}$
    $\widehat{A}_{ij} = \frac{\sum_{t=1}^{T-1} \mathbb{E}_q[h_t=j, h_{t+1}=i]}{\sum_{t=1}^{T-1} \mathbb{E}_q[h_t=j]}$
end while

- $\mathbb{E}_q[h_t] = \alpha(h_t)\beta(h_{t+1})/Z$
- $\mathbb{E}_q[h_t, h+1] = p(h_t, h_{t+1}|x_{1:T}) \propto p(x_{1:t}, h_t)p(h_{t+1}|h_t)p(x_{t+1}|h_{t+1})p(x_{t+2:T}|h_{t+1}) = \alpha(h_t)p(h_{t+1}|h_t)p(x_{t+1}|h_{t+1})\beta(h_{t+1}) = \alpha(h_t)AO[:, h_{t+1}]\beta(h_{t+1})$

Table of Contents

Dynamic Time Warping

Hidden Markov Models
- Inference
- Learning
- Decoding
- HMM Applications
- HMM Variants

Decoding in HMMs

- The precise inference question / La question précise pour l'inférence.

$$p(x_{1:T}, h^*_{1:T}|\theta) = \max_{h_{1:T}} p(x_{1:T}, h_{1:T}|\theta)$$

Decoding in HMMs

- The precise inference question / La question précise pour l'inférence.

$$
\begin{aligned}
p(x_{1:T}, h^*_{1:T}|\theta) &= \max_{h_{1:T}} p(x_{1:T}, h_{1:T}|\theta) \\
&= \max_{h_{1:T}} \prod_{t=1}^{T} p(x_t|h_t)p(h_t|h_{t-1})
\end{aligned}
$$

Decoding in HMMs

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$$

- This is a huge max! / C'est une operation immense.

Decoding in HMMs

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\end{aligned}
$$

- This is a huge max! / C'est une operation immense.
- What can we do? / Qu'est-ce qu'on peut faire?

The Dynamic Programming Solution (Again)

$$V(h_t) = p(x_t|h_t) \max_{h_{t-1}} p(h_t|h_{t-1}) \underbrace{p(x_{t-1}|h_{t-1}) \ldots \underbrace{p(x_2|h_2) \max_{h_1} p(h_2|h_1)p(x_1|h_1) \underbrace{p(h_1)}_{V(h_1)}}_{V(h_2)}}_{V(h_{t-1})}$$

We run this recursion, and then backtrack to find the optimal path $h^*_{1:T}$. (The Viterbi algorithm) / On roule la recurrence et puis backtrack pour trouver la trace optimale $h^*_{1:T}$.

Table of Contents

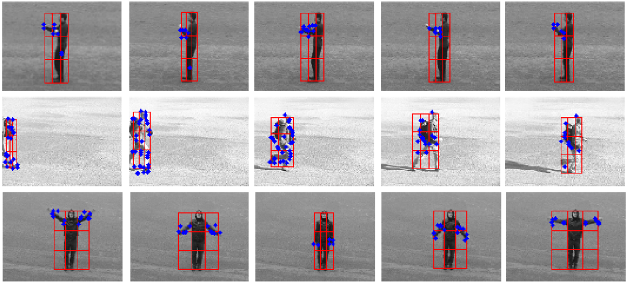
Dynamic Time Warping

Hidden Markov Models

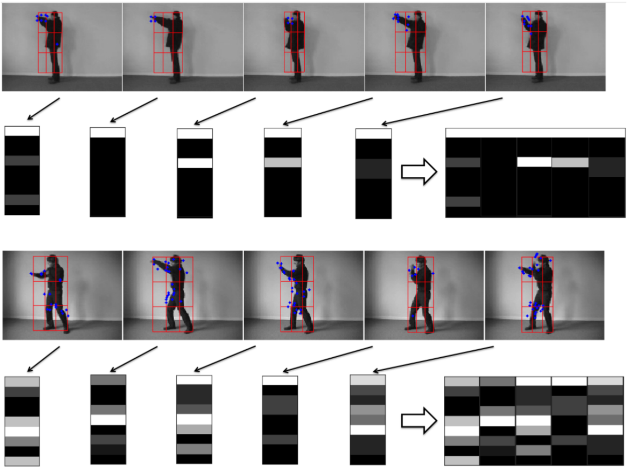
- Inference
- Learning
- Decoding
- HMM Applications
- HMM Variants

An HMM Learning Application

- Human Action Recognition



Getting the sequences



HMMs for classification

- Train an HMM for each class / On entraine un HMM pour chaque classe.
- In test time we assign to the HMM that yields the max likelihood.

$$\widehat{c}_n = \arg\max_k p(x_n|\theta_k)$$

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- We saw this type of thing before, remember? / Vous vous souvenez de ça?

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- We saw this type of thing before, remember? / Vous vous souvenez de ça?
- Generative classification.. B=boxing, HC=Hand Clapping, HW=Hand Waving ...

EM, 70.1 %

	B	HC	HW	J	R	W
B	**32**	4	1	0	1	0
HC	1	**31**	6	0	1	0
HW	0	1	**29**	0	0	0
J	1	0	0	**17**	20	3
R	0	0	0	7	**10**	0
W	2	0	0	12	4	**33**

HMMs for speech recognition

- Each state should correspond to a semantically meaningful thing. (e.g. a phoneme) / Chaque état HMM devrait correspondre à quelque chose qui a une sense semantique.
- Someone saying 'one'. / Quelqu'un qui dit 'one'.

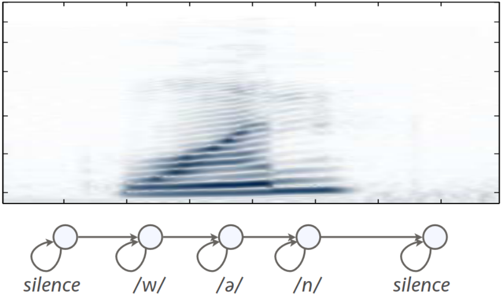


Image taken from UIUC MLSP course.

HMM learning on speech

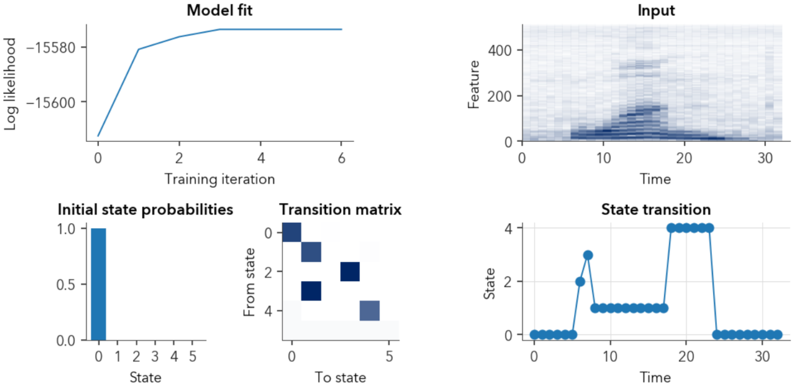


Image taken from UIUC MLSP course.

HMM learning on speech

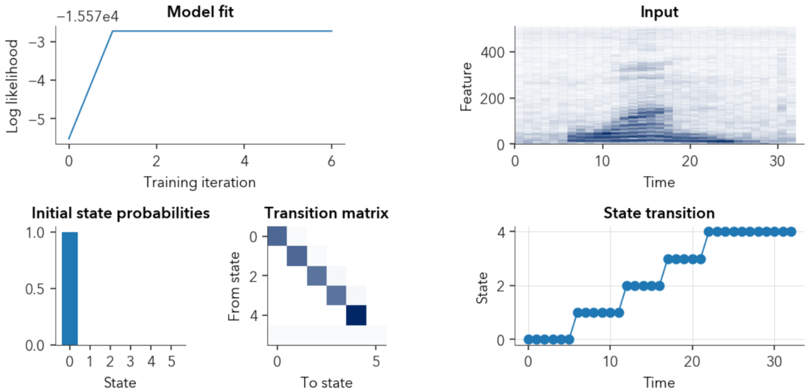


Image taken from UIUC MLSP course.

A big ASR system with HMMs

- A real life HMM speech recognizer in a nut shell / La sommaire d'un système ASR avec HMMs.

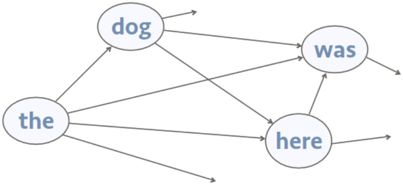


Image taken from UIUC MLSP course.

- You have an HMM for each word, then you connect the HMMs..

And don’t say HMMs are outdated!

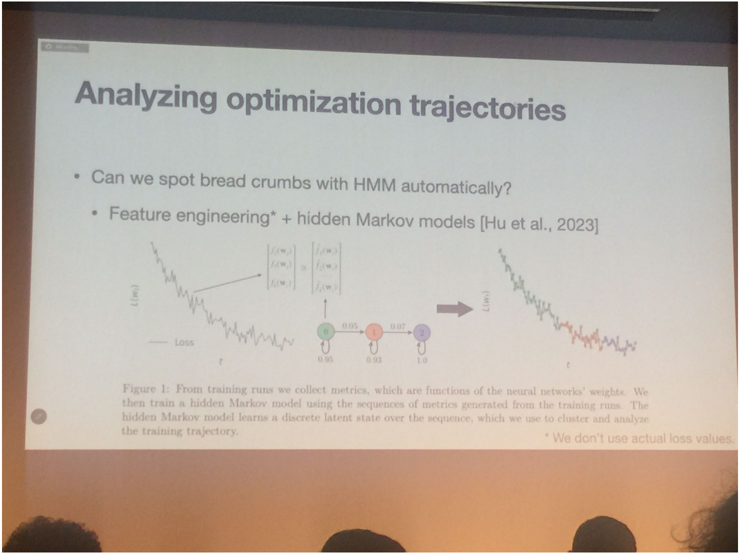


Image taken when I was at SANE 2023 last year

Using HMMs to analyze the training behavior of LLMs from 2023.

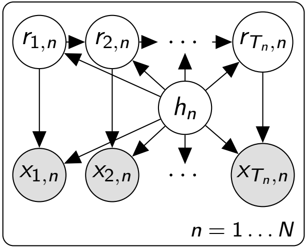
Table of Contents

Dynamic Time Warping

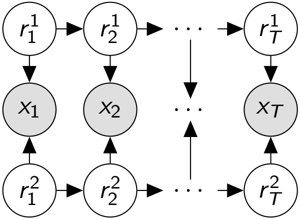
Hidden Markov Models
- Inference
- Learning
- Decoding
- HMM Applications
- HMM Variants

More Advanced HMM Variants

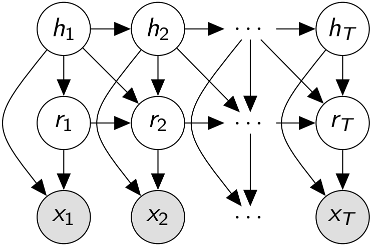
- Mixture of HMMs



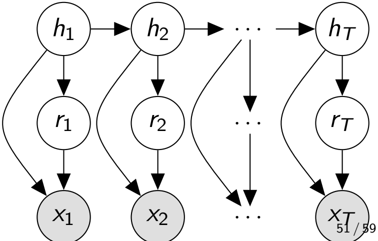
- Factorial HMM



- Switching HMMs

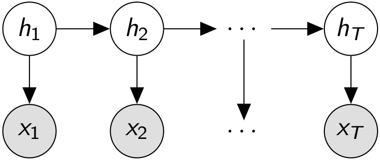


- HMM with Mixture observations



Super Similar but different: Linear Dynamical System

- Model:



$$
\begin{aligned}
h_n | h_{n-1} &\sim \mathcal{N}(h_n; A h_{n-1}, \Sigma_1) \\
x_n | h_n &\sim \mathcal{N}(x_n; O h_n, \Sigma_2)
\end{aligned}
$$

- $h_n \in \mathbb{R}^K$, latent variables/variables latents.
- $x_n \in \mathbb{R}^L$, observed data items / données observées.
- $O \in \mathbb{R}^{L \times K}$, the emission matrix / la matrice d'emission, $A \in \mathbb{R}^{K \times K}$, the transition matrix / la matrice de transition.
- $\theta = \{O, A\}$ parameters / paramètres.
- The inference is done with a Kalman filter. / Le fameux filtre de Kalman est utilisé pour l'inférence.

Tired of directed graphs? MRFs

- The joint distribution is defined with clique "potentials".

$$p(h_{1:K}, x_{1:J}|\theta) = \frac{1}{Z(\theta)} \prod_{C \in \mathcal{G}} \exp(\theta^T \phi(x_C, h_C))$$

Tired of directed graphs? MRFs

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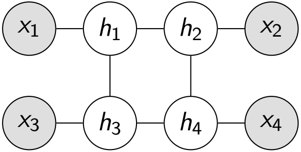
$$p(h_{1:K}, x_{1:J}|\theta) = \frac{1}{Z(\theta)} \prod_{C \in \mathcal{G}} \exp(\theta^T \phi(x_C, h_C))$$

- Example: (An image segmentation model)



$$\begin{aligned}\phi(x_C, h_C) &= \phi_1(h_i, h_{\mathcal{N}(i)}) + \phi_2(x_i, h_i) \\ &= \theta_1 1_{[h_i = h_{\mathcal{N}(i)}]} + \theta_2 1_{[h_i \neq h_{\mathcal{N}(i)}]} \\ &\quad + \sum_{l,k} \theta_{3,i,k} 1_{[x_i = l][h_i = k]}\end{aligned}$$

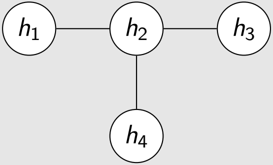
$$Z(\theta) = \int \prod_{C \in \mathcal{G}} \exp(\theta^T \phi(x_C, h_C)) dx_{1:J} dh_{1:K}$$

The notorious partition function!

How to do inference in general graphs?

- Forward-Backward algorithm is an instance of "Belief Propagation".

Example

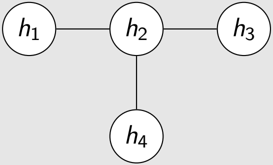


$$p(h_{1:4}) = \frac{1}{Z}\psi(h_1, h_2)\psi(h_2, h_4)\psi(h_2, h_3)$$

$$p(h_2) \propto \sum_{h_1,h_3,h_4} \psi(h_1, h_2)\psi(h_2, h_4)\psi(h_2, h_3)$$
$$= \underbrace{\left(\sum_{h_1} \psi(h_1, h_2)\right)}_{\mathrm{m}_{1\to 2}} \underbrace{\left(\sum_{h_4} \psi(h_2, h_4)\right)}_{\mathrm{m}_{4\to 2}} \underbrace{\left(\sum_{h_3} \psi(h_2, h_3)\right)}_{\mathrm{m}_{3\to 2}}$$

Example continued

Example



$$p(h_{1:4}) = \frac{1}{Z}\psi(h_1, h_2)\psi(h_2, h_4)\psi(h_2, h_3)$$

$$\begin{aligned}
p(h_1) &\propto \sum_{h_2, h_3, h_4} \psi(h_1, h_2)\psi(h_2, h_4)\psi(h_2, h_3) \\
&= \sum_{h_2} \psi(h_1, h_2) \left(\sum_{h_4} \psi(h_2, h_4) \right) \left(\sum_{h_3} \psi(h_2, h_3) \right) \\
&= \sum_{h_2} \psi(h_1, h_2) \mathrm{m}_{4\to 2}(h_2) \mathrm{m}_{3\to 2}(h_2)
\end{aligned}$$

BP, summarized

- Compute all messages for all possible (i, j) pairs with,

$$\mathrm{m}_{i\to j}(h_j) = \sum_{h_i} \psi(h_i, h_j) \overbrace{\prod_{l\in\mathcal{N}(i)\setminus j} \mathrm{m}_{l\to i}(h_i)}^{\text{Incoming Messages to node } i}$$

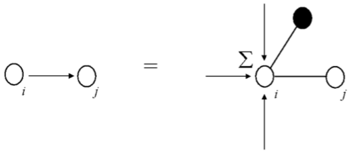


Figure is taken from Yedidia et al. 2001.

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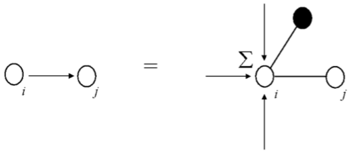

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- The Belief for node i is $B(h_i) = p(h_i) = \prod_{j\in\mathcal{N}(i)} \mathrm{m}_{j\to i}(h_i)$.

BP, summarized

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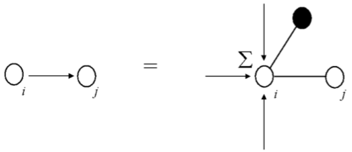

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- One pass from leaves to root and one pass from leaves to root, and we are done.

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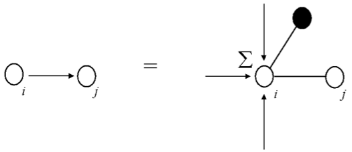

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- The Belief for node i is $B(h_i) = p(h_i) = \prod_{j \in \mathcal{N}(i)} m_{j \to i}(h_i)$.
- One pass from leaves to root and one pass from leaves to root, and we are done.
- BP converges to true beliefs in trees. What about general graphs?

Recap

- Dynamic Time Warping
  - It's a way to measure distances between sequences. / Way to measure distances between sequences
- HMMs
  - Probability distribution over sequences. Can be thought of generalization of DTW. / HMMs définissent une distribution de probabilité sur les séquences. Vous pouvez le voir comme étant une géneralization de DTW.
- HMM Applications
  - Speech Recognition, Human Action Recognition, Sequence Clustering, LLM State Transition Understanding, ...
- More Advanced HMMs
  - Mixture of HMMs, Swiching HMMs, HMMs with Mixture observations, Linear Dynamical Systems, MRFs, ...

Suggested reading

- The classic HMM Tutorial:
  `http://www.cs.ubc.ca/~murphyk/Bayes/rabiner.pdf`
- Bishop chapter 13.

Next week

- Graph Signal Processing / Machine Learning