

IFT 4030/7030,
Machine Learning for Signal Processing
**Week6: Machine Learning 3,
Classification**

Cem Subakan



UNIVERSITÉ
LAVAL



Mila

- Le devoir 1 est dû la semaine prochaine.
 - ▶ Homework 1 was due this friday, now it's due next friday.
- Aujourd'hui: Classification

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Classification Intro

Generative Classification

Discriminative Classification

- Linear Classifiers

- The perceptron algorithm

- Logistic Regression

Non-Linear Classification

- Kernel Logistic Regression

- Neural Network Classification

Supervised Learning

- So far we have done unsupervised learning to discover structures.
 - ▶ Jusqu'à maintenant on a juste fait de l'apprentissage non-supervisé.
Le but était de découvrir la structure.
- Now, we will do classification / detection, or supervised learning in other words.
 - ▶ Maintenant on va faire de l'apprentissage supervisé.

A simple example



A simple example



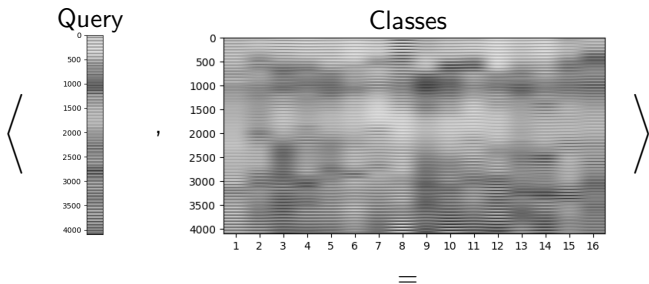
How can we assign the query to a class? / Comment peut-on assigner l'exemple en question à une classe?

We can simply calculate inner products!

We can calculate inner products! / On peut calculer des produits scalaires!

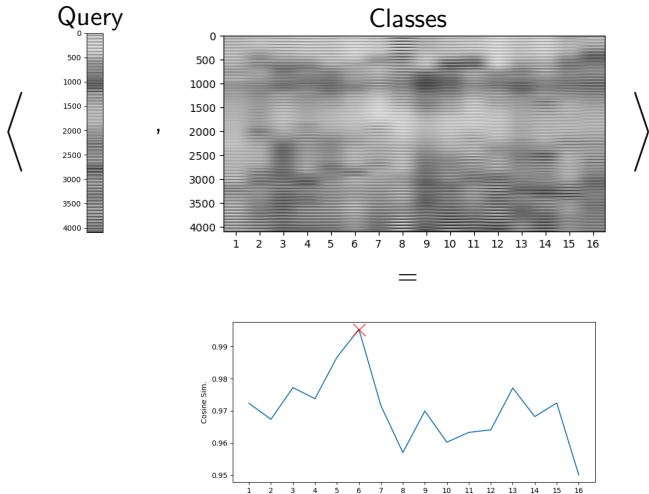
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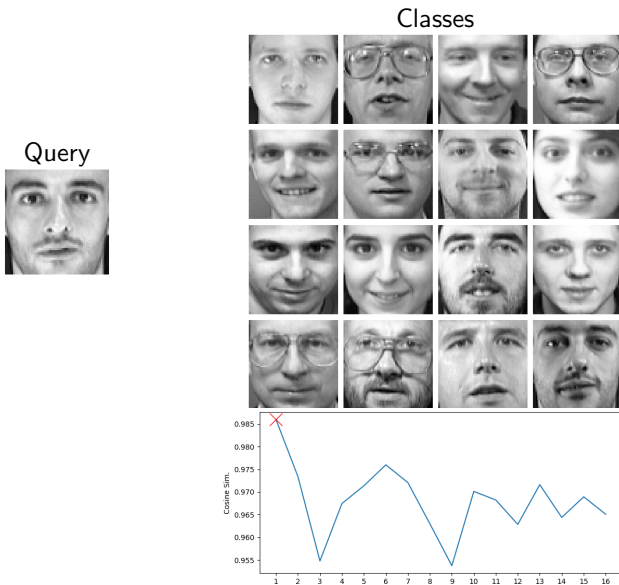
It worked but will it this time?



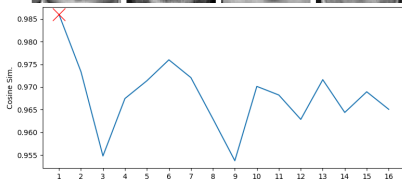
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It worked but will it this time?



Didn't work this time! / Ça pas fonctionné!

What can we do to fix this?

- We will take a statistical approach (as usual). This will consider the class distributions. / On va prendre une approche de statistique comme d'habitude qui va prendre en compte les distributions de classes.

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- Approach 1: Generative Classification
 - ▶ Approche 1: Classification Générative
 - ▶ We will learn a distribution over samples in the class. / On va fitter une distribution sur les échantillons dans la classe.
- Approach 2: Discriminative Classification
 - ▶ Approche 2: Classification Discriminative
 - ▶ We will learn how the class distribution separate. / On va fitter une fonction pour comprendre comment les classes se séparent.

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 - ▶ Approche 2: Classification Discriminative
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- We will build step by step towards neural nets, and why we need them in this lecture. / On va builder graduellement à pourquoi on a besoin des réseaux de neurones dans ce cours.
 - ▶ Linear Classifiers (Perceptron Algo. Logistic Regression, Kernel Methods, and then Multilayer perceptron)

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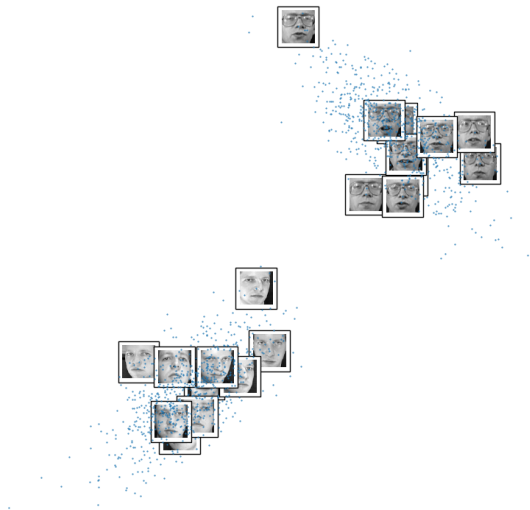
Non-Linear Classification

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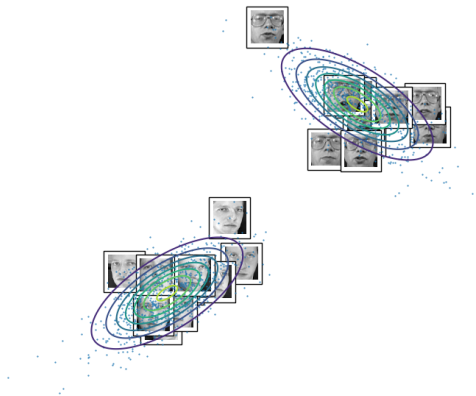
Generative Classification

- Generative Classification fits distributions to each class /
Classification générative fit une distribution à chaque classe.



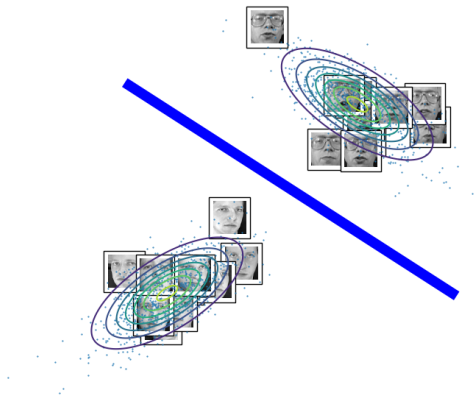
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Generative Classification

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Generative Classification

- Training Time: Fit a distribution $p(x|\theta_k)$ to each class k with maximum likelihood.
 - ▶ L'entraînement: On va fitter une distribution $p(x|\theta_k)$ pour chaque classe k avec maximum likelihood.

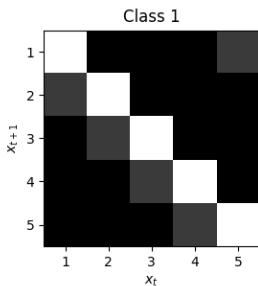
$$\max_{\theta_k} \sum_{n \in \text{class } k} \log p(x_n|\theta_k)$$

- Test Time: Evaluate the likelihood for each model. Assign to the largest likelihood class!
 - ▶ L'entraînement: Évaluez le likelihood pour chaque modèle. Assignez à la classe avec le likelihood plus grand.

$$\hat{c} = \arg \max_k \log p(x_{\text{test}}|\theta_k)$$

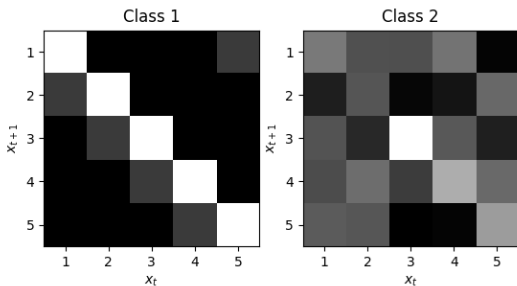
Example Application: Classifying Sequences

- class 1 \sim Markov(A_1)
- class 2 \sim Markov(A_2)



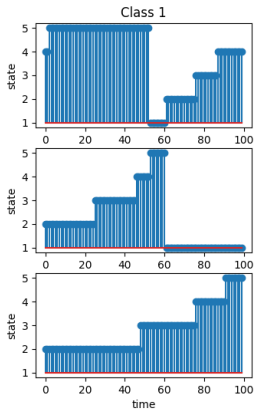
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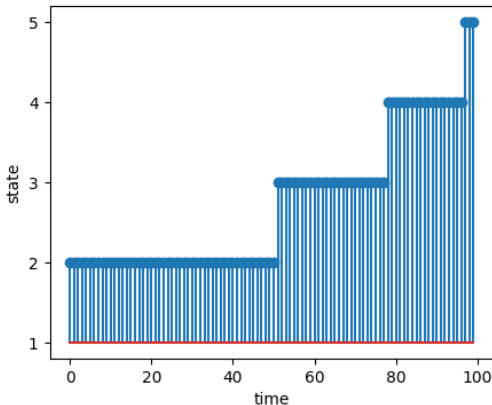
Example Application: Classifying Sequences

- class 1 \sim Markov(A_1)
- class 2 \sim Markov(A_2)



Test time

- Here's a test sequence / Une séquence de test



- Do you think it belongs to which class? / Vous pensez que cette séquence appartient à quelle classe?

$$\log p(x_{1:T} | A_k, \pi_k) = \log \left(p(x_1 | \pi_k) \prod_{t=2}^T p(x_t | x_{t-1}, A_k) \right)$$

Test time

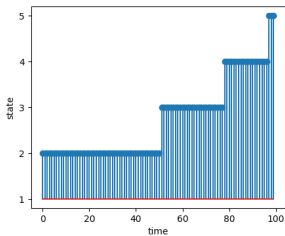
$$\begin{aligned}\log p(x_{1:T}|A_k, \pi_k) &= \log \left(p(x_1|\pi_k) \prod_{t=2}^T p(x_t|x_{t-1}, A_k) \right) \\ &= \log p(x_1|\pi_k) + \sum_{t=2}^T \log p(x_t|x_{t-1}, A_k)\end{aligned}$$

Test time

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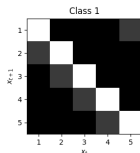
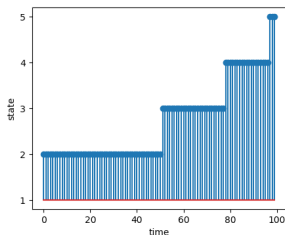
Now, let's calculate

- Observation sequence / Séquence observé



Now, let's calculate

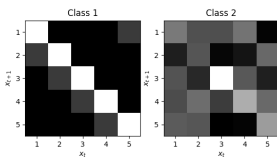
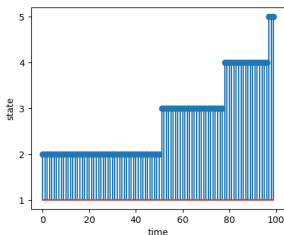
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$$\begin{aligned}\log p(x_{1:T}|A_1, \pi_1) &= \log \left(\pi_1 \prod_{t=2}^T A_1(x_t, x_{t-1}) \right) \\ &= \log \left(\frac{1}{5} \cdot 0.95 \cdot 0.95 \cdot 0.95 \dots \right) \\ &= -15.52\end{aligned}$$

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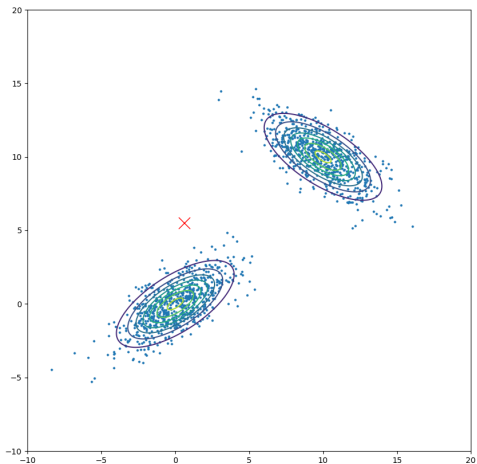


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$$\begin{aligned}\log p(x_{1:T}|A_2, \pi_2) &= \log \left(\pi_1 \prod_{t=2}^T A_2(x_t, x_{t-1}) \right) = \log \left(\frac{1}{5} \cdot 0.21 \cdot 0.21 \cdot 0.21 \dots \right) \\ &= -117.93\end{aligned}$$

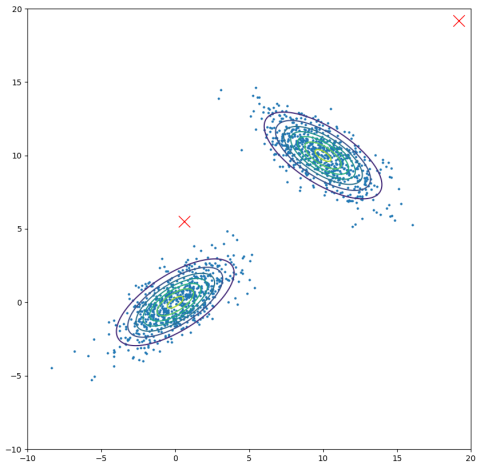
Let's do something simpler

Which class should we assign the red point? / Quel classes doit-on assigner au point rouge?



How about this?

How about this second point? / Et le deuxième point?



The principled way to derive the decision boundaries

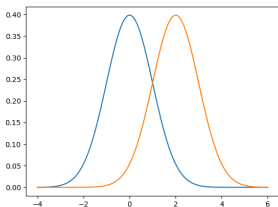
- The discriminant function / La fonctionne de discrimination

$$d(x) = \log \frac{p(x|c = 1)}{p(x|c = 2)} = \log \frac{p(x|\theta_1)}{p(x|\theta_2)}$$

- The decision boundary: $d(x) = 0$.
- For Gaussians / simple densities this can be derived analytically.

Deriving the decision boundary

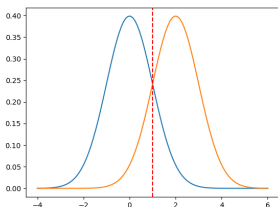
- Derive the boundary in 1d case / Dérivons dans le cas 1d.



$$d(x) = \log \frac{p(x|\theta_1)}{p(x|\theta_2)} = \log \frac{\mathcal{N}(0; 1)}{\mathcal{N}(2; 1)}$$

Deriving the decision boundary

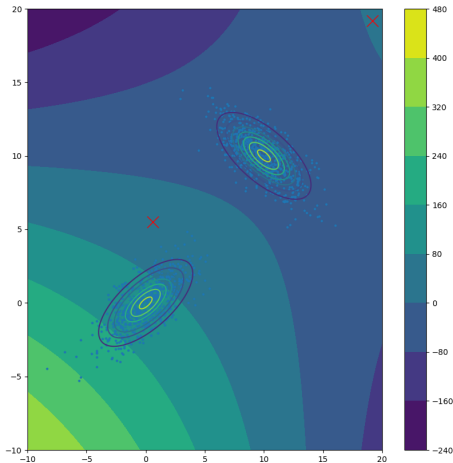
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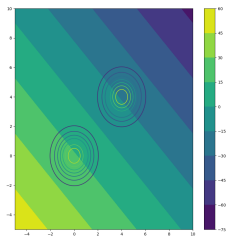
$$\begin{aligned}d(x) &= \log \frac{p(x|\theta_1)}{p(x|\theta_2)} = \log \frac{\mathcal{N}(0; 1)}{\mathcal{N}(2; 1)} \\ &= \log \frac{x^2}{(x-2)^2} = 0 \\ \rightarrow x^2 &= (x-2)^2 \rightarrow x = 1\end{aligned}$$

The 2d decision boundary

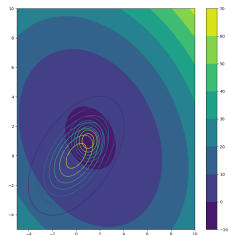
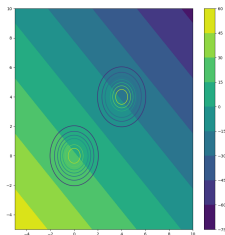
$$\text{Plot } d(x) = \log \frac{\mathcal{N}([x,y]^T; \mu_1, \Sigma_1)}{\mathcal{N}([x,y]^T; \mu_2, \Sigma_2)}$$



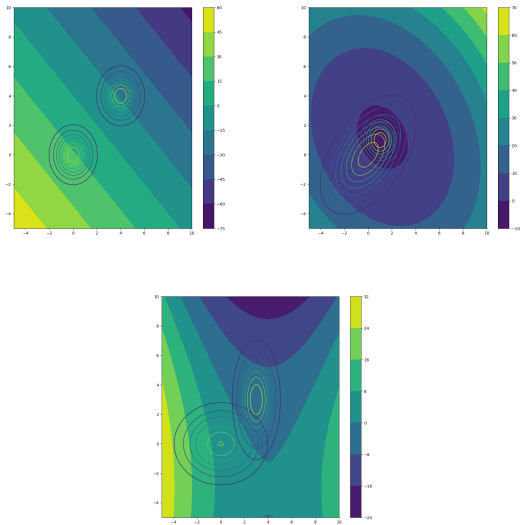
Few other cases



Few other cases



Few other cases



All these cases can be derived analytically! / C'est possible analytiquement
calculer

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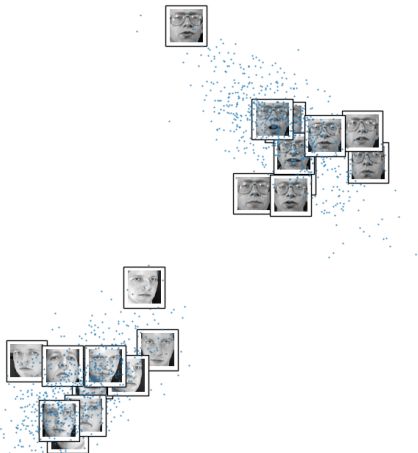
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Discriminative Classification

- Discriminative classification directly learns a decision boundary / Classification discriminative apprend directement un borne de décision.



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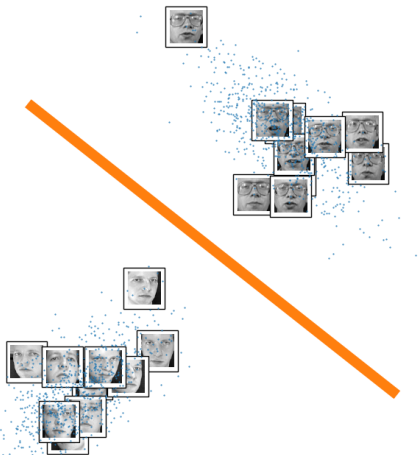


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$$w^\top x \geq 0 \text{ if } c = 1$$

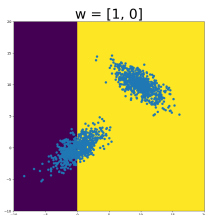
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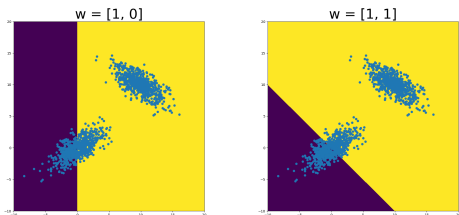


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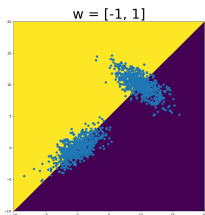
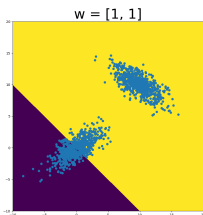
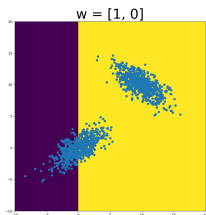


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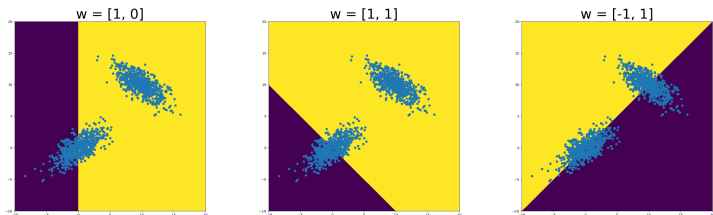


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- Btw, we will also add a bias term so that $f(x) = w^\top x + b$ / En passant on va aussi ajouter un biais.

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 - ▶ If $\text{sgn}(w^\top x_n) = c_n$ do nothing.

The perceptron algorithm

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- The perceptron algorithm.
 - ▶ If $\text{sgn}(w^\top x_n) = c_n$ do nothing.
 - ▶ If $\text{sgn}(w^\top x_n) = -c_n$, then $w = w + \eta c_n x_n$. (η is a learning rate)

The perceptron algorithm

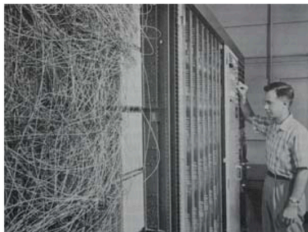
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 - ▶ Do these updates until convergence. / On répète jusqu'qu'on converge.

The perceptron algorithm

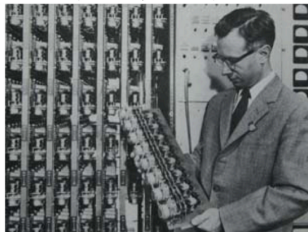
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 - ▶ Do these updates until convergence. / On répète jusqu'qu'on converge.
 - ▶ Note that $c_n \in \{-1, 1\}$. / Notez que $c_n \in \{-1, 1\}$.

The perceptron

Feature extraction processor

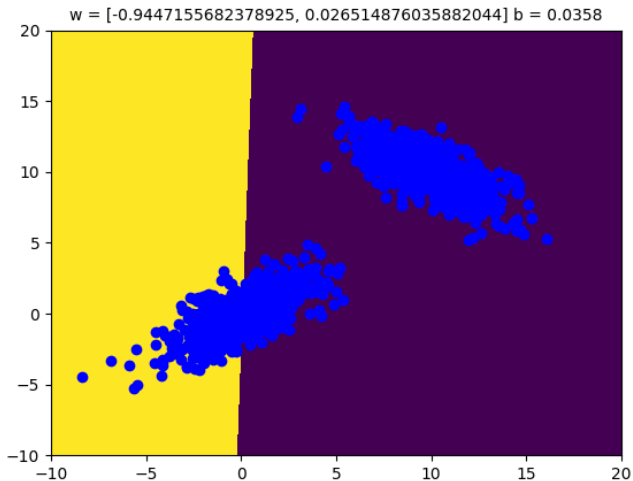


*Perceptron weights
(motor driven potentiometers)*

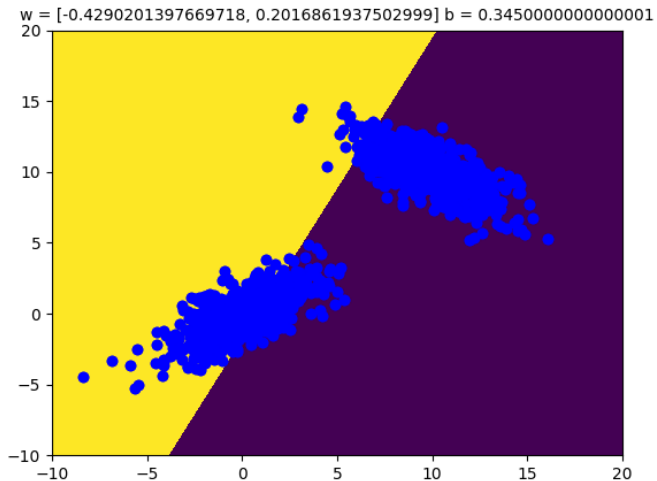


Taken from UIUC MLSP class

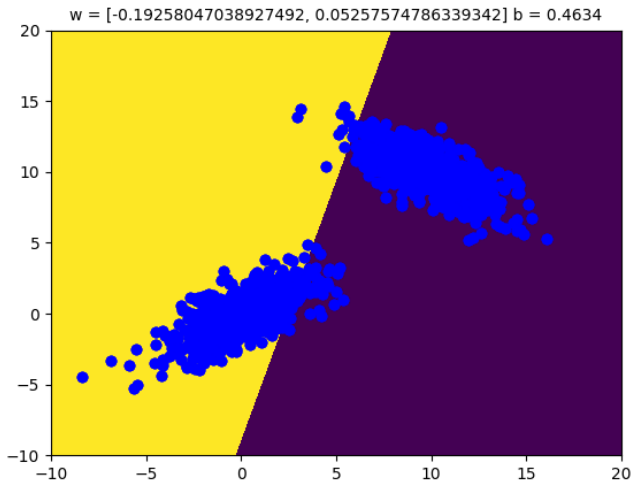
Perceptron Epochs



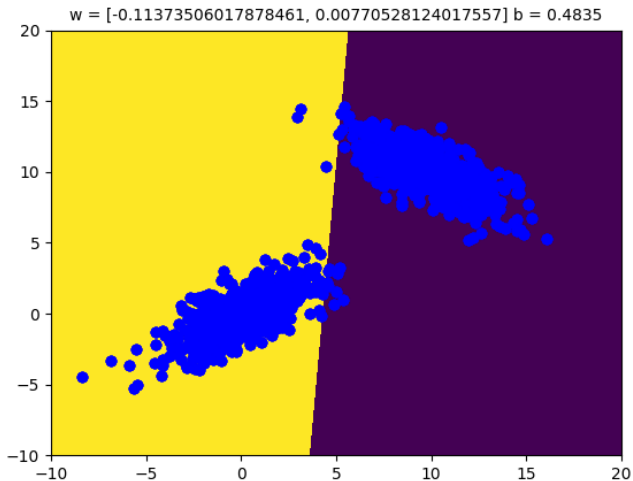
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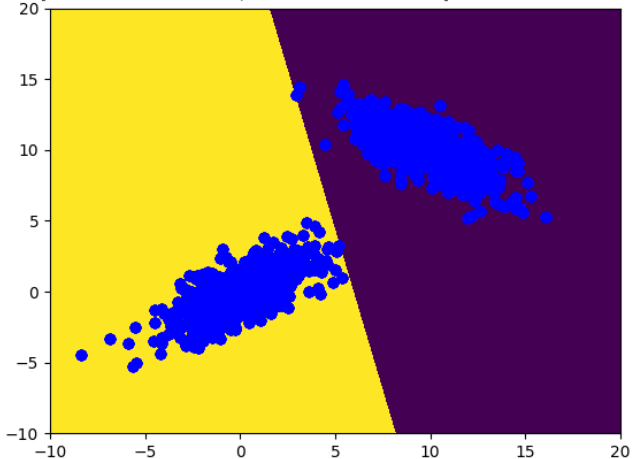


Perceptron Epochs



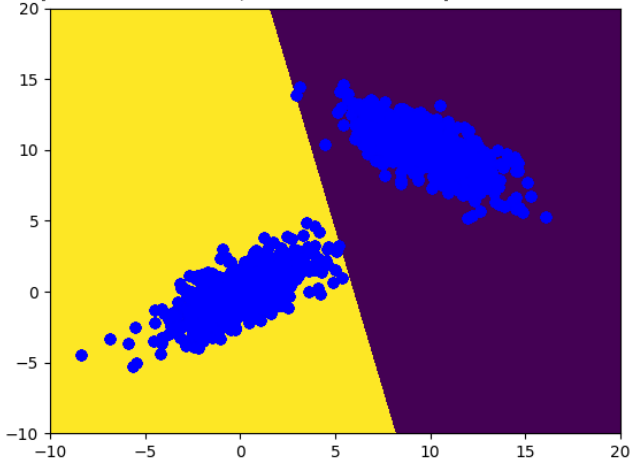
Perceptron Epochs

$w = [-0.08226628225110869, -0.01821149067324113]$ $b = 0.4884999999999998$



Perceptron Epochs

$w = [-0.08226628225110869, -0.01821149067324113]$ $b = 0.4884999999999998$



Deriving the perceptron algorithm

- Define a cost function / Définissons une fonction de cout:

$$\mathcal{L}(w, b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n (w^\top x_n + b)$$

Deriving the perceptron algorithm

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$$\mathcal{L}(w, b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n (w^\top x_n + b)$$

- Do the gradient descent updates: / Faisons les GD updates:

$$w = w - \eta \nabla_w \mathcal{L}(w, b)$$

$$b = b - \eta \nabla_b \mathcal{L}(w, b)$$

Deriving the perceptron algorithm

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$$w = w - \eta \nabla_w \mathcal{L}(w, b)$$

$$b = b - \eta \nabla_b \mathcal{L}(w, b)$$

- The gradient: / Le gradient:

$$\nabla_w \mathcal{L}(w) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n x_n$$

$$\nabla_b \mathcal{L}(b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n$$

Deriving the perceptron algorithm

- Define a cost function / Définissons une fonction de cout:

$$\mathcal{L}(w, b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n (w^\top x_n + b)$$

- Do the gradient descent updates: / Faisons les GD updates:

$$w = w - \eta \nabla_w \mathcal{L}(w, b)$$

$$b = b - \eta \nabla_b \mathcal{L}(w, b)$$

- The gradient: / Le gradient:

$$\nabla_w \mathcal{L}(w) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n x_n$$

$$\nabla_b \mathcal{L}(b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n$$

- The updates / Les updates: (Same as the perceptron algo! / Les memes updates qu'algo perceptron!)

$$w = w + \eta \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n x_n$$

$$b = b + \eta \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n$$

But can we do something better?

- Perceptron is basically / Perceptron est simplement:

$$f(x) = u(w^T x + b)$$

where $u(\cdot)$ is a step function (or $\text{sign}(\cdot)$ depending on how we define c_n) . / où $u(\cdot)$ est un 'step function'.

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 - ▶ À cause de la step function on ne peut pas utiliser la modèle directement dans le fonctionne de cout.

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- Because of that we can not directly use above within a loss function, and instead we use a modified objective. (Not all items get updated.)
 - ▶ À cause de la step function on ne peut pas utiliser la modèle directement dans le fonctionne de cout.
- What if we want to have smooth, differentiable transitions? / Et si on veut avoir des transitions plus smooth?

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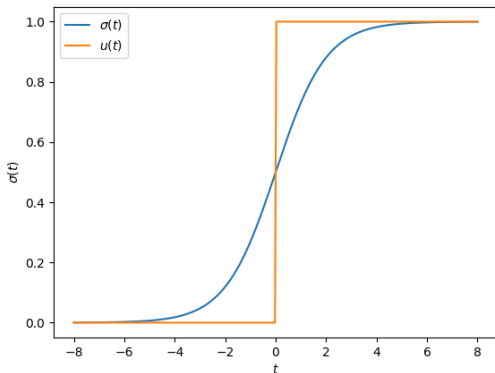
Non-Linear Classification

Kernel Logistic Regression

Neural Network Classification

The logistic function

- We can use the logistic function $\sigma(t) = \frac{1}{1+e^{-t}}$. / On peut utiliser la fonction logistique.



Logistic regression

- Set the estimator as $f(w, b) = \sigma(w^\top x + b)$.

Logistic regression

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- Same neural network, different activation function. / La meme réseau de neurones, different fonctionne d'activation.

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Logistic regression

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- Same neural network, different activation function. / La meme réseau de neurones, different fonctionne d'activation.
- What will be the loss function? / Quelle sera la fonctionne de coût?
- How about the negative Bernoulli log-likelihood?

Bernoulli Distribution

- $BE(y; \pi) = \pi^y(1 - \pi)^{1-y}$, $y \in \{0, 1\}$.
- Let's take the log / Prenons le logarithme

$$\log BE(y; \pi) = y \log \pi + (1 - y) \log(1 - \pi)$$

- We will parametrize the Bernoulli distribution with $\pi = w^\top x + b$ /
On est en train de paramétriser la distribution Bernoulli.

Logistic regression loss function

- Training loss / La fonction de cout pour l'entrainement:

$$\begin{aligned}\mathcal{L}(w, b) &= \sum_n (-y_n \log \pi_w(x) + (1 - y_n) \log(1 - \pi_w(x))) \\ &= \sum_n \left(-y_n \log \left(\sigma(w^\top x + b) \right) - (1 - y_n) \log \left(1 - \sigma(w^\top x + b) \right) \right)\end{aligned}$$

- Let's calculate the gradient with respect to w . / Calculons le gradient par rapport à w .

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- The chain rule: $\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = f'(g(x)) \cdot g'(x)$

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- The chain rule: $\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = f'(g(x)) \cdot g'(x)$
- Few more things: $\frac{\sigma(x)}{dx} = \sigma(x)(1 - \sigma(x))$, $\frac{\log x}{dx} = 1/x$.

The gradient wrt w

- Just do it! / Faisons-le!

$$\begin{aligned}\frac{d\mathcal{L}(w, b)}{dw} &= - \sum_n y_n \frac{\sigma(w^\top x_n + b)}{\sigma(w^\top x_n + b)} (1 - \sigma(w^\top x_n + b)) x_n \\ &\quad + \sum_n (1 - y_n) \frac{1 - \sigma(w^\top x_n + b)}{1 - \sigma(w^\top x_n + b)} \sigma(w^\top x_n + b) x_n\end{aligned}$$

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- Makes a lot of sense! / Ça fait du sense!

The gradient wrt w

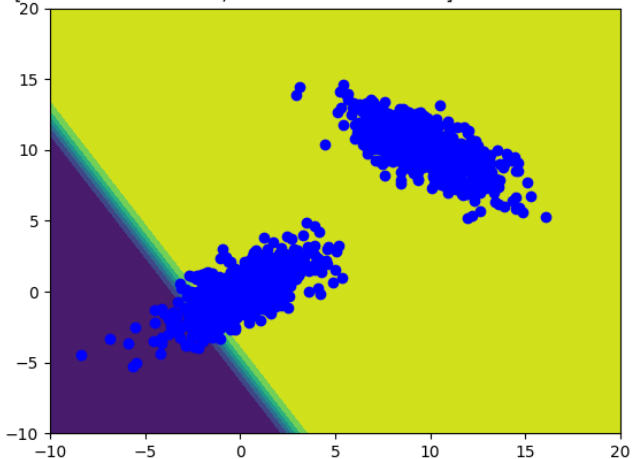
- Just do it! / Faisons-le!

$$\begin{aligned}\frac{d\mathcal{L}(w, b)}{dw} &= - \sum_n y_n \frac{\sigma(w^\top x_n + b)}{\sigma(w^\top x_n + b)} (1 - \sigma(w^\top x_n + b)) x_n \\ &\quad + \sum_n (1 - y_n) \frac{1 - \sigma(w^\top x_n + b)}{1 - \sigma(w^\top x_n + b)} \sigma(w^\top x_n + b) x_n \\ &= - \sum_n y_n \frac{\cancel{\sigma(w^\top x_n + b)}}{\cancel{\sigma(w^\top x_n + b)}} (1 - \sigma(w^\top x_n + b)) x_n \\ &\quad + \sum_n (1 - y_n) \frac{\cancel{1 - \sigma(w^\top x_n + b)}}{\cancel{1 - \sigma(w^\top x_n + b)}} \sigma(w^\top x_n + b) x_n \\ &= \sum_n (\sigma(w^\top x + b) - y_n) x_n\end{aligned}$$

- Makes a lot of sense! / Ça fait du sens!
- torch will automatically do this for us. However, one needs to do this at least once in their life!
 - ▶ torch fait ça automatiquement mais on doit faire cette exercice au moins une fois dans notre vie!

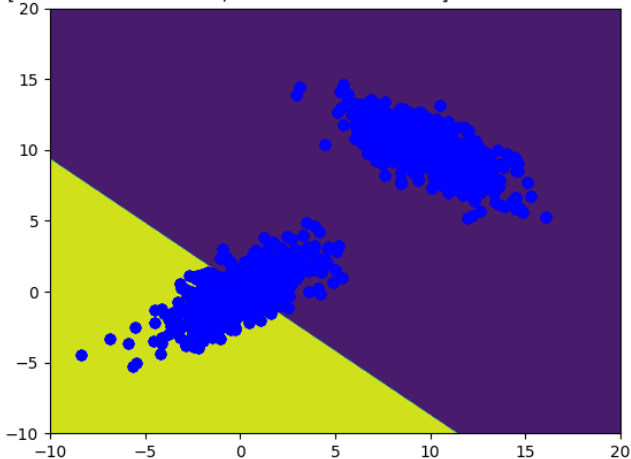
Logistic Regression Epochs

$w = [2.774843454360962, 1.6030927896499634]$ $b = 4.142136573791504$



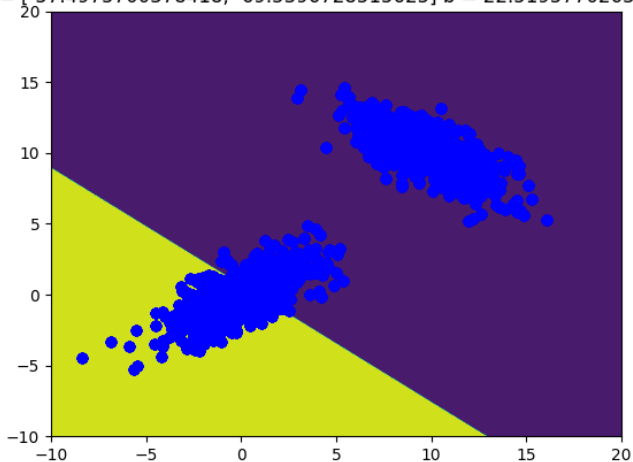
Logistic Regression Epochs

$w = [-64.31231689453125, -71.11492156982422]$ $b = 10.29466533660888$



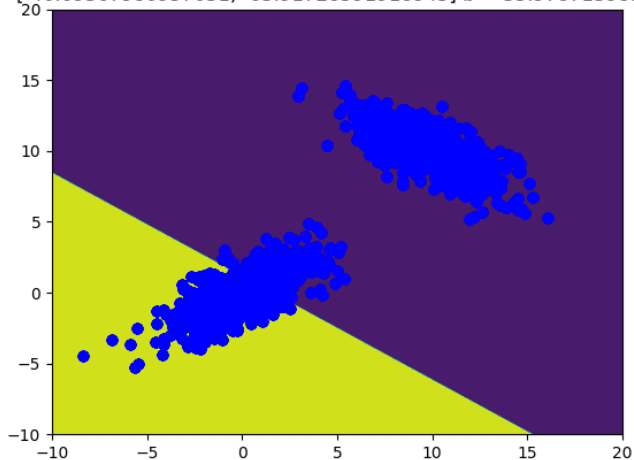
Logistic Regression Epochs

$w = [-57.4975700378418, -69.5396728515625]$ $b = 22.519577026367188$



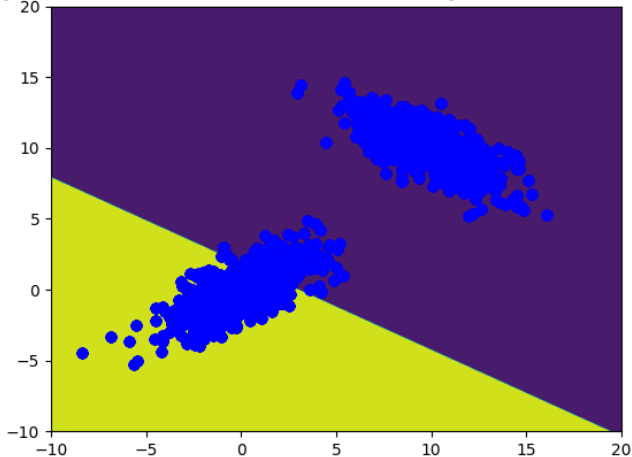
Logistic Regression Epochs

$w = [-46.69367980957031, -63.91728591918945]$ $b = 35.9787139892578$:



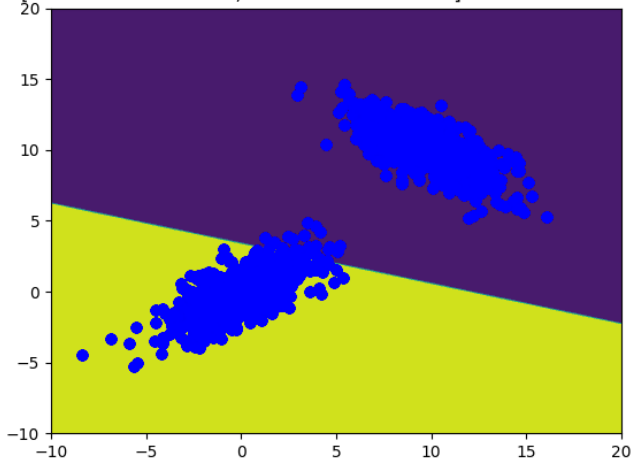
Logistic Regression Epochs

$w = [-32.589866638183594, -53.49893569946289]$ $b = 48.29494476318359$



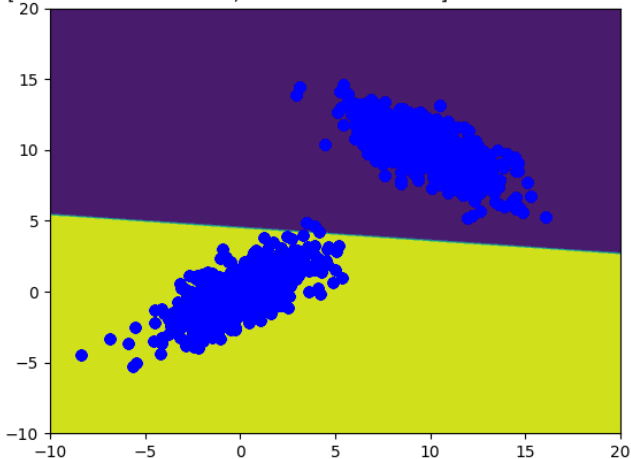
Logistic Regression Epochs

$w = [-9.948322296142578, -35.09688186645508]$ $b = 59.3541374206543$



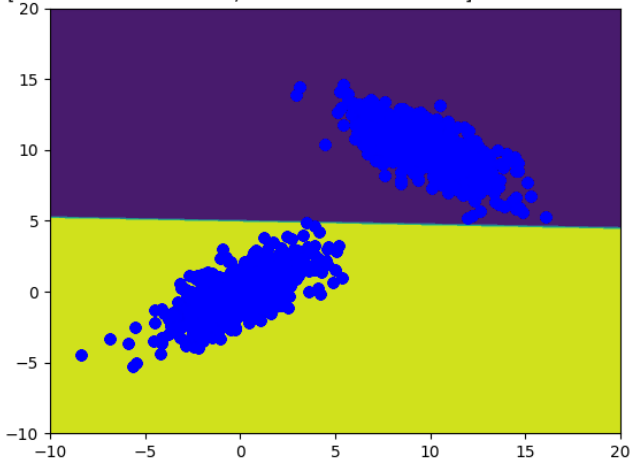
Logistic Regression Epochs

$w = [-2.5054750442504883, -27.3735294342041]$ $b = 61.56295776367187$



Logistic Regression Epochs

$w = [-0.6339982151985168, -24.971982955932617]$ $b = 62.072887420654$



Generalizing to multiway classification

- Define (a new network!) / On définit une nouvelle réseau

$$\pi_k := \exp(w_k^\top x + b_k) / \sum_l \exp(w_l^\top x + b_l)$$

Generalizing to multiway classification

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$$\pi_k := \exp(w_k^\top x + b_k) / \sum_l \exp(w_l^\top x + b_l)$$

- The loss function is now going to be the log-likelihood of discrete distribution / La fonction de coût sera maintenant le log-likelihood de la distribution discrète.

$$\mathcal{L}(W, b) = -\log \left(\prod_n \text{Discrete}(y_n; \pi_{1:K}) \right)$$

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Non-Linear Classification

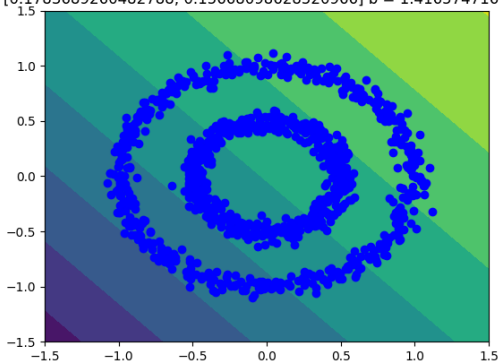
- Kernel Logistic Regression

- Neural Network Classification

Non-Linear Classification

- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?

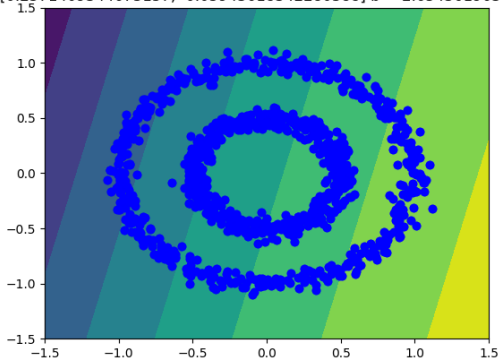
$n = [0.1783689260482788, 0.15668098628520966]$ $b = 1.41657471656799$



Non-Linear Classification

- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?

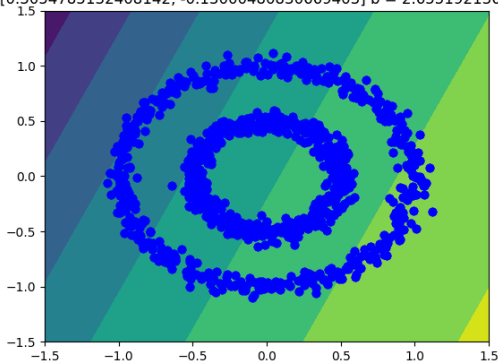
$v = [0.2571409344673157, -0.05943618342280388]$ $b = 2.65436196327209$



Non-Linear Classification

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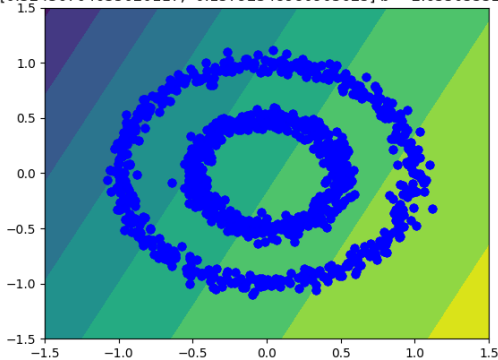
$v = [0.3054785132408142, -0.13000480830669403]$ $b = 2.65519213676452$



Non-Linear Classification

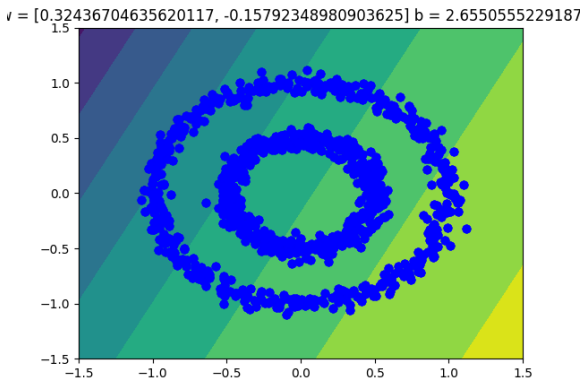
- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?

$v = [0.32436704635620117, -0.15792348980903625]$ $b = 2.6550555229187$



Non-Linear Classification

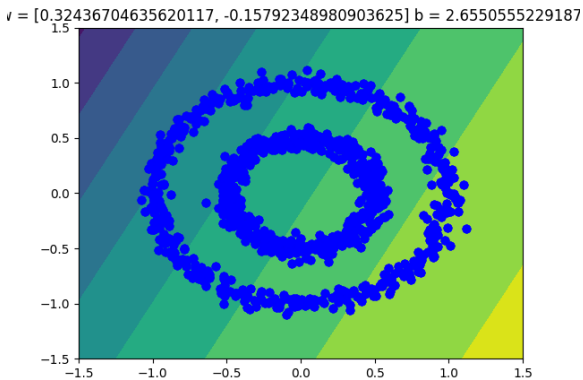
- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?



- Linear classifier is not enough! / Classificateur linéaire n'est pas suffisante!

Non-Linear Classification

- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?



- Linear classifier is not enough! / Classificateur linéaire n'est pas suffisante!
- What can we do? / Qu'est-ce qu'on peut faire?

The Kernel Trick

- We can extend our linear classifier $y = w^\top x \rightarrow y = w^\top \phi(x)$. We introduce a feature transformation $\phi(\cdot)$.
 - ▶ On améliore notre classificateur linéaire en introduisant une transformation de feature $\phi(\cdot)$.

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- As we talked about it last week for Kernel-PCA, introducing $\phi(\cdot)$ is not possible if ϕ is large dimensional. / Comme on en a parlé $\phi(\cdot)$ n'est pas réalisable si on map a une large nombre de dimensions.

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- **Kernel Trick:** We can substitute/ On substitue $w = \sum_n a_n \phi(x_n)$,
so

$$\begin{aligned} f(x) &= \sigma \left(\sum_n a_n \phi(x)^\top \phi(x_n) \right) \\ &= \sum_n \sigma(a_n k(x, x_n)) \end{aligned}$$

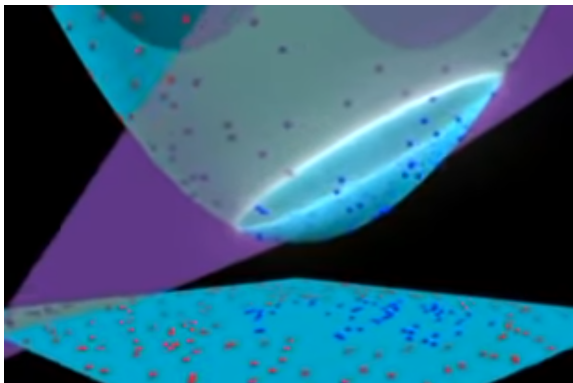
The Kernel Trick

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- We can incorporate this in many places. But let's look at injecting this idea in logistic regression.
 - ▶ On peut incorporer cette idée dans différents place. Mais d'abord essayons d'injecter cette idée dans la regression logistique.

Motivating the Kernel Trick



Let's watch!

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Kernel Logistic Regression

- The coin bias is now estimated with the kernel-trick / Le biais est maintenant estimé avec le kernel trick.

$$\pi = \sigma \left(\sum_n a_n k(x, x_n) \right)$$

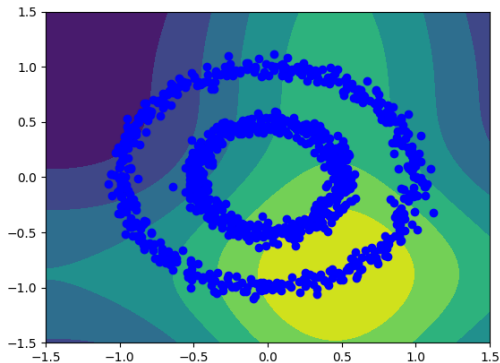
- Then the loss becomes, / le fonctionne de cout devient,

$$\begin{aligned} \mathcal{L}(a) = \sum_n & \left(-y_n \log \left(\sigma \left(\sum_j a_j k(x_n, x_j) \right) \right) \right. \\ & \left. - (1 - y_n) \log \left(1 - \sigma \left(\sum_j a_j k(x_n, x_j) \right) \right) \right) \end{aligned}$$

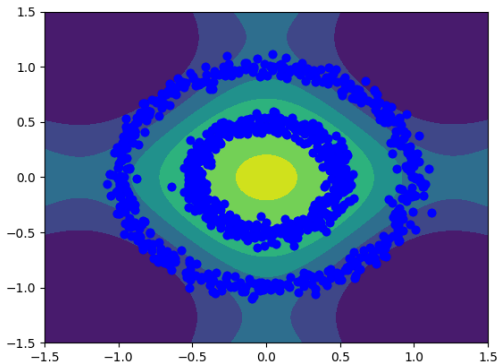
- Kernel functions are chosen from options such as RBF Kernel, Polynomial Kernel,.. / On choisit le kernel entre les options qu'on est habitué.

$$k_{\text{rbf}}(x, x_n) = \exp(-\gamma \|x - x_n\|_2)$$

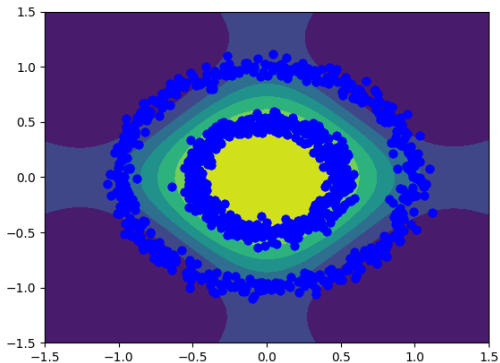
Kernel Logistic Regression Learning Steps



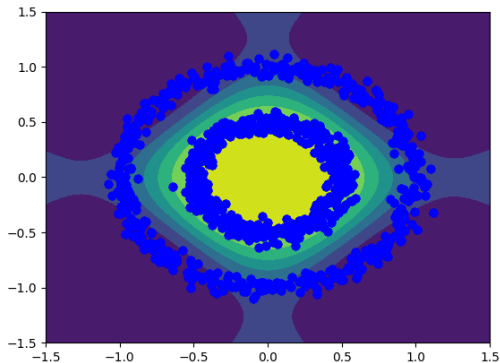
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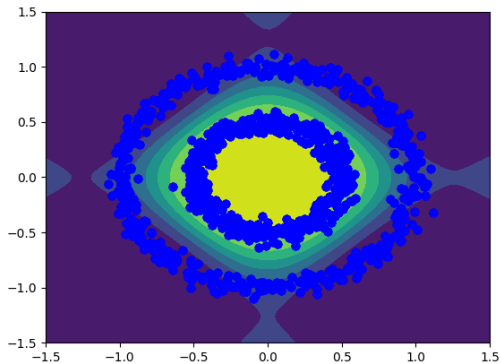
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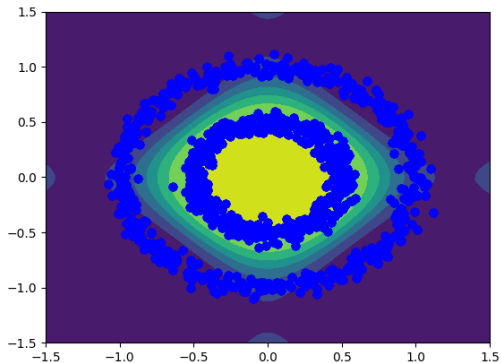
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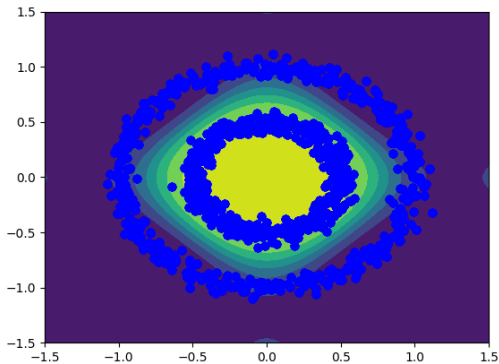
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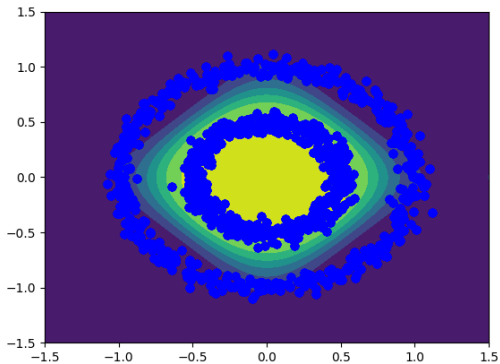
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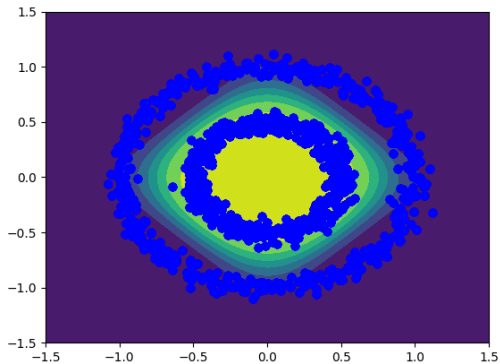
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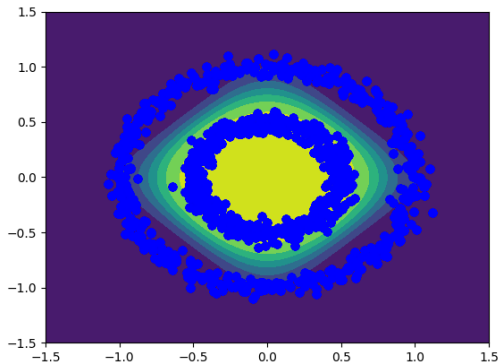


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Non-Linear Classification

- Kernel Logistic Regression

- Neural Network Classification**

Neural Network Classification

- The bias parameter can be estimated with a general function $f(x)$.

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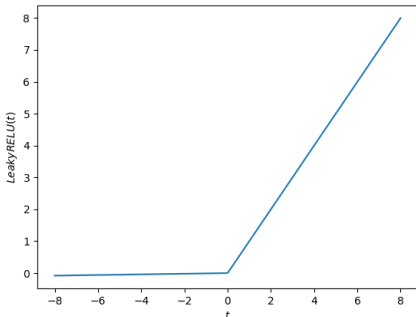
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- The loss function becomes / La fonction de cout devient:

$$\mathcal{L}(\theta) = \sum_n \left(-y_n \log \left(\sigma \left(f_\theta(x_n) \right) \right) - (1 - y_n) \log \left(1 - \sigma \left(f_\theta(x_n) \right) \right) \right)$$

An Example Neural Network

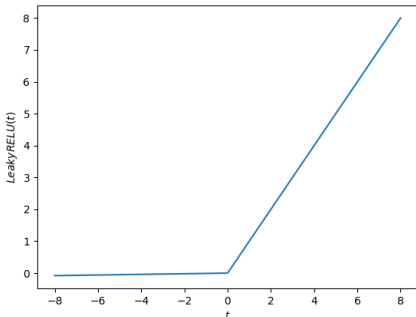
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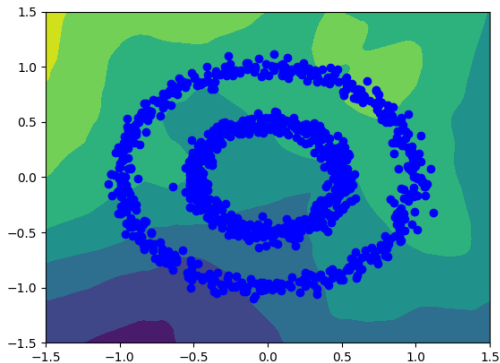
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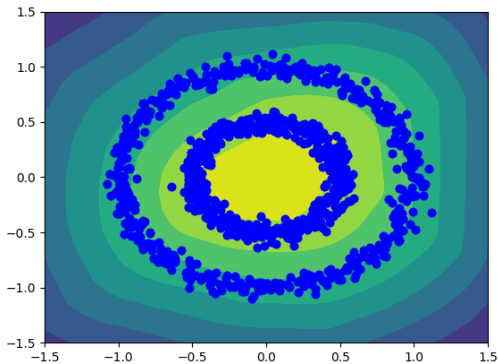


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- Let's say we define $\phi_{W_1}(x) := h(W_1^T x)$.

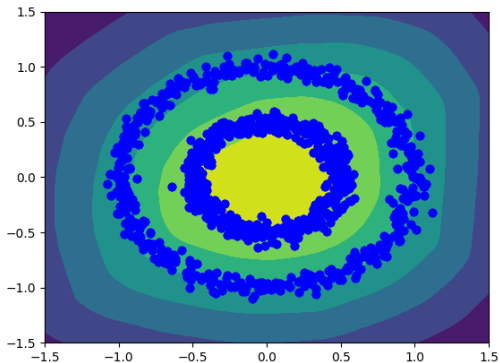
Neural Network Classifier Learning Steps



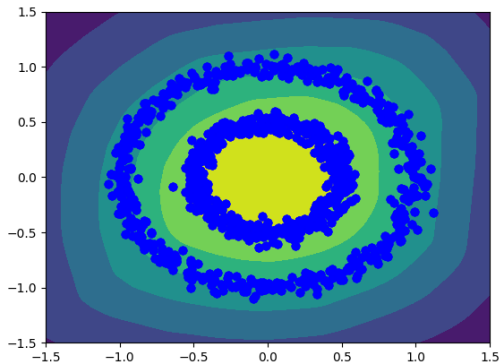
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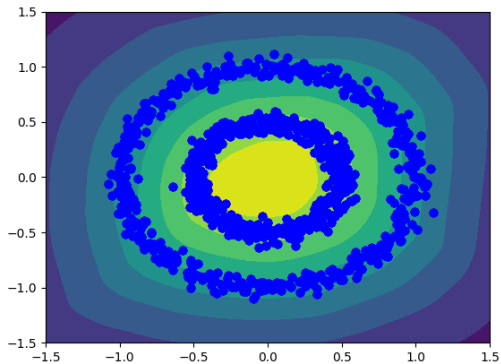
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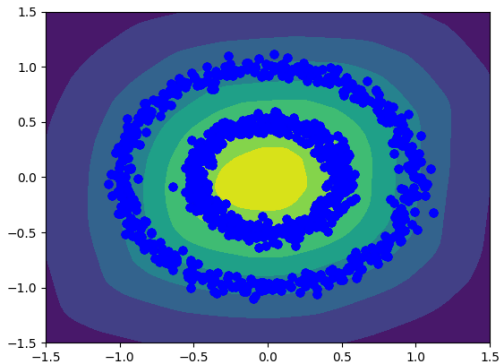
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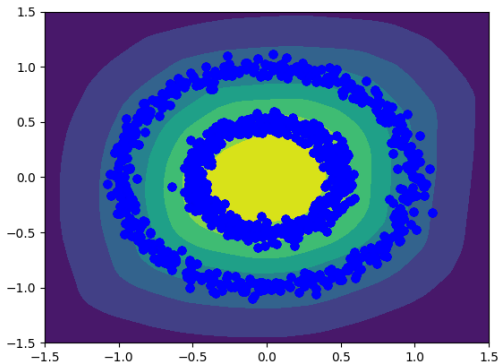
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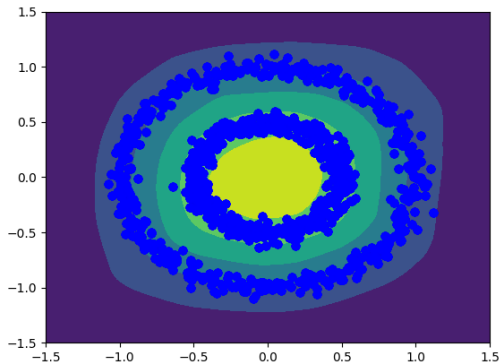
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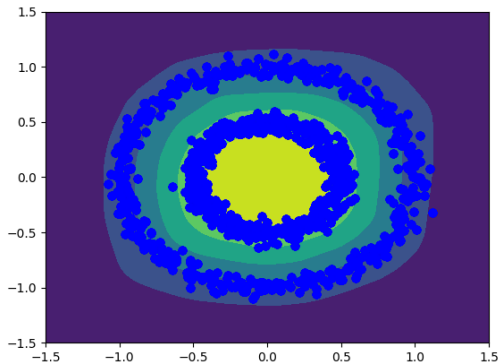
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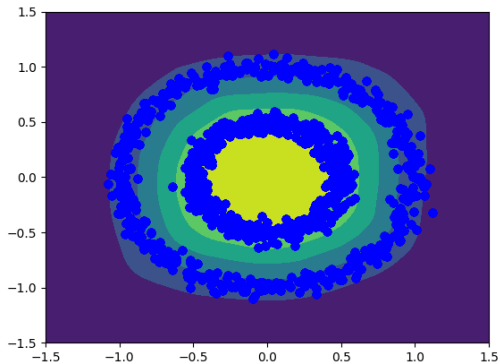
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Kernel Methods vs Neural Networks

- Kernel Methods are not very suitable for large datasets because of the hard dependency on the training set x_1, \dots, x_N . / Les méthodes de noyau ne sont pas très adaptées pour des datasets larges parce qu'ils dépendent sur le training set.
- Kernel Methods come with an interpretability advantage. At the end of the day it's a linear model, and we have the importance weights. / Les méthodes de noyaux sont plus interprétables car ils sont linéaires.
- Neural Nets are more modern because of the way we train them. They are more amenable to train with SGD. / Les réseaux de neurones sont plus adaptés à entraîner avec SGD sur des grands jeux de données.

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- We have not talked about SVMs. It's an important idea to know, but you can get that from many other classes. On n'a pas parlé de SVMs.

Suggested reading

- Bishop, chapters 4, 5, 6

Next week

- More deep learning!