IFT 4030/7030,
Machine Learning for Signal Processing
Week1: Class Intro,
Linear Algebra Refresher

Cem Subakan

What is this class?

- What do you think this class is?

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- What do you think this class is?
- Is it a Machine Learning class?
- Is it a Signal Processing class?

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- What do you think this class is?
- Is it a Machine Learning class?
- Is it a Signal Processing class?
- What is Machine Learning?
- What is Signal Processing?

Signal Processing

- Here's the wikipedia definition:

  Signal processing is an electrical engineering subfield that focuses on analyzing, modifying and synthesizing *signals*, such as sound, images, potential fields, seismic signals, altimetry processing, and scientific measurements.[1] Signal processing techniques are used to optimize transmissions, digital storage efficiency, correcting distorted signals, subjective video quality and to also detect or pinpoint components of interest in a measured signal.[2]

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- Hm, this kinda sounds like machine learning.

How are signals different than data?

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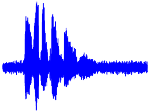
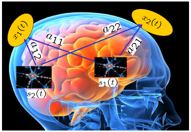
- So, signals are just data?
- Yeah-(ish).
- Why are we calling them signals then?
- When we speak of signals, we refer more to structured data. (Order matters)
- And, saying ‘signals’, ‘signal processing’ implies a more Electrical Engineering way to the approach.

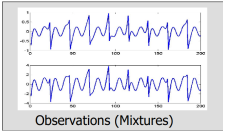
Example Signals

- Images, Audio/Speech

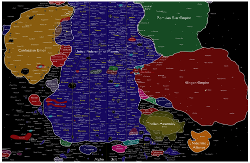


- Brains





- Financial Time Series, Graphs



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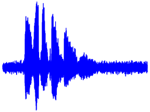

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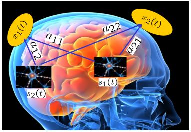


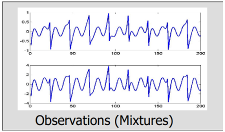

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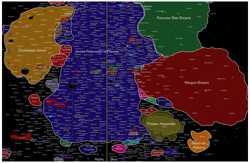

- More?

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But why bother? Isn't ML what's hip now?

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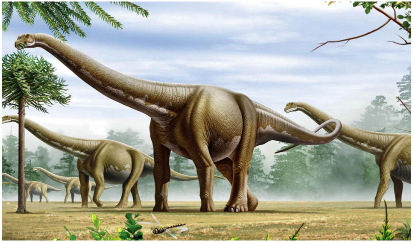
- What about signal processing, doesn't that cover what we need?
  - No!



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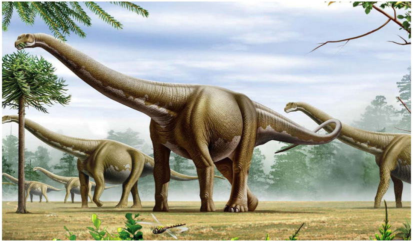

  - Traditional SP is typically **NOT** statistical, doesn't handle the statistical patterns of the signal well.
  - Traditional SP: Filtering, acquision, analog-digital-analog conversion, transmission
  - There is statistical signal processing also, but it doesn't go much beyond adaptive filtering.

MLSP: Machine Learning for Signal Processing

- How to build systems that would work with sequences and solve machine intelligence tasks on them?
  - Various tasks with Speech and Audio: ASR, Speech Enhancement, Music Transcription...

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  - Financial Time Series Prediction
  - Understanding Biomedical Sequences
  - Generating Videos

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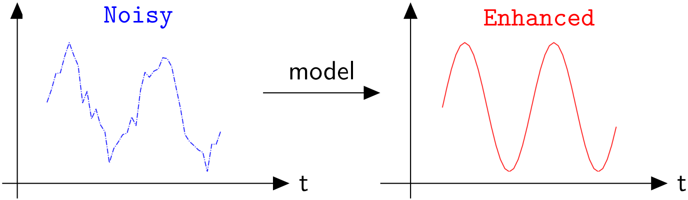
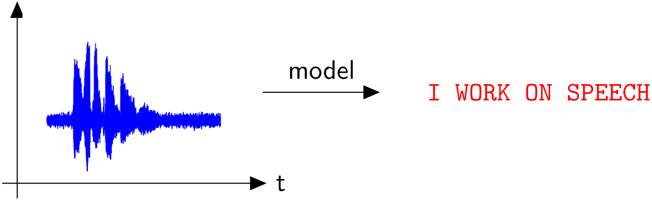
  - Various tasks with Speech and Audio: ASR, Speech Enhancement, Music Transcription...
  - Financial Time Series Prediction
  - Understanding Biomedical Sequences
  - Generating Videos
  - More...

Speech and Audio Modeling

- Speech Enhancement

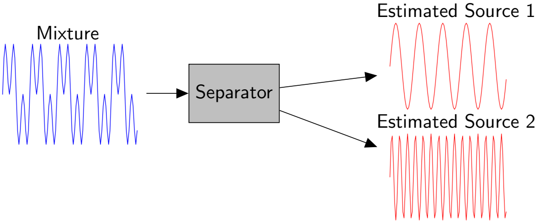


- Speech Recognition

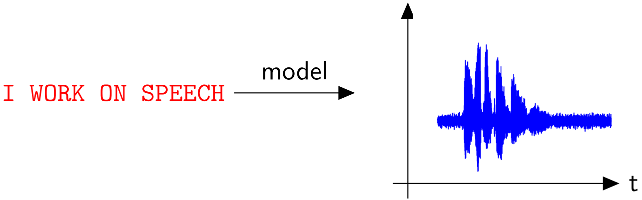


Speech and Audio Modeling

- Speech Separation

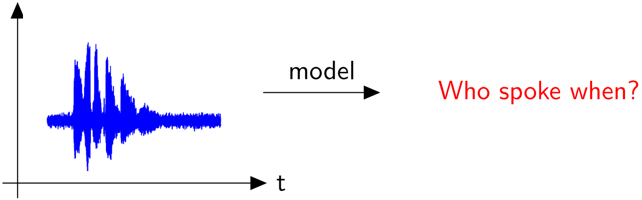


- Text-to-Speech

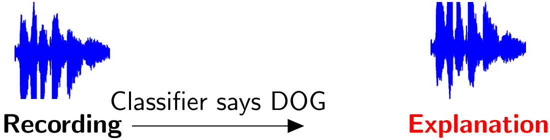


Speech and Audio Modeling

- Speaker Diarization



- Neural Network Explanation



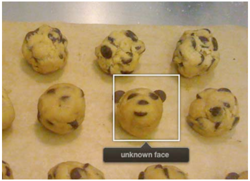
- Other problems: Generating Deep fakes, Detecting deep fakes, Music Source Separation, Music Transcription, Sound Event Detection/Classification...

Speech and Audio Modeling

- Field with huge economic value & job opportunities,
  - Speech Recognition (e.g. Siri)
  - Speech Enhancement (e.g. Google meet, Zoom)
  - Text-to-Speech
  - Speaker Verification, Spoof Detection(Banks)
  - Speaker Diarization for Meeting Analysis (Nuance, Microsoft)
  - Source Separation (e.g. Beatles Rock Band, Meeting Analysis)

Other real-life applications

- Face recognition



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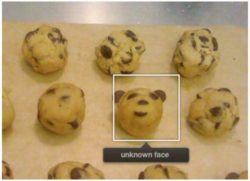

- Brain-machine interfaces

Other real-life applications

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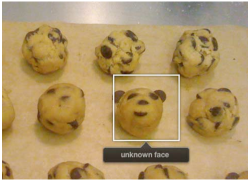

- Brain-machine interfaces

- Real time bio-signal analysis, learning generative models for bio/medical signals, condition monitoring (mining machines, production machines), Stock market, many more..

About this class

- This is class heavy on practice. How do we make things that work?
- We do not do deep theory in this class.
  - We will not prove things.
  - We will not stay Keras level either.
  - Our goal is to give useful insights, be useful.
- We go fast, our typical lecture could be a class.

Syllabus: Basics

- Linear Algebra
  - This class
- Probability
  - Probability Calculus, Random Variables, Bayesian vs Frequentist Principles
- Signal Processing
  - Signal Representations, Fourier Transform, Sampling

Syllabus: Machine Learning

- Decompositions
  - PCA, NMF, Linear Regression, Tensor Decompositions
- Classification
  - Logistic Regression, Maximum Margin, Kernels, Boosting
- Deep Learning
  - Deep Learning Firearms, Pytorch, Julia
- Optimization
  - Convex optimization
  - Gradient Descent and friends
  - Non-Convex optimization
- Clustering
  - Kmeans, Spectral Clustering, DBScan
- Unsupervised Non-linear learning
  - Manifold Learning, Deep Generative Models
- Time Series Models
  - HMMs, Kalman Filters

Syllabus: Fun Stuff

- Speech Recognition
- Speech Enhancement/Separation
- Text-to-speech
- Representation Learning Methods for Sequences
- Generative Models for Sequences
- Text prompted models (text prompted image / sound generation)
- Neural Network Interpretation Methods
- Graph Signal Processing / ML

Evaluation

- Homeworks (45%)
  - 3 homeworks, you need to work on these alone!
  - I would like you to typeset math in LaTeX. So if you don't know it, start learning it!
  - Do not use Generative AI, if you want to learn!
  - You will need to code. But we will reward good quality presentation of results.
- Weekly Labs (10%)
  - You will work on hands-on application of the things we talk about. TAs will lead the online sessions.
- Final Project (45%)

Final project

- This will be a mini-conference.
- Each paper will receive 3 peer-reviews (from you). We will evaluate the quality of your reviews (5% of your 45% project grade).
- You will work in teams of 2-3 (no more, no less)
- We will ask who did what in the project. So no freeriding!
- Start making friends!
- Mid-October, proposals are due
- Last 1-2 weeks, paper deadline.

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- Start making friends!
- Mid-October, proposals are due
- Last 1-2 weeks, paper deadline.
  - We will accept all the papers, and you will make a presentation.
  - However, you need to do a good job to get a good grade.
  - If it's a good paper, we can also work together to submit it to a real conference! We can work together towards that.

Communications

- We will have teams page where will have a forum, and you will submit your assignments.
- Be active on the forum, ask questions. Find friends for the project.
- We will do the announcements on teams, so sign-up for it!
- Check `https://ycemsubakan.github.io/mlsp.html` for class material.

Instructor: Who am I?

- **Instructor:** Cem Subakan
  - cem.subakan@ift.ulaval.ca
  - Assistant Prof. in Computer Science,
    Mila Associate Academic Member.
  - Just send me a message you if you want to meet.
- I work on machine learning for Speech and Audio.
  - Interpretability
  - Speech Separation & Enhancement
  - Multi-Modal Learning
  - Continual Learning
  - Probabilitic Machine/Deep Learning
- I review for many major conferences, involved in the organization of several MLSP workshops.
- I have written a lot of papers involving MLSP topics, worked with many people, also saw the industry side of things.

Who are the TAs?

- Sara Karami
  - sara.karami.1@ulaval.ca
- Mathieu Bazinet
  - mabaz21@ulaval.ca
- TAs will hold the online lab sessions (Fridays 15h00-16h50)
- The office hours will be on fridays (the second half of the lab sessions)
- Advice:
  - If you need help do not bombard them at the last minute. Seek help early.

Who are you?

- Name, department, grad/undergrad?
  - What are your interests?
  - Hint: Take notes, and contact the person if something picks your interest.

Table of Contents

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Scalars, Vectors, Matrices, Tensors

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- Matrix, x of size $L \times M$
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- Tensor, x of size $L \times M \times N$
$$x = \begin{matrix} \begin{bmatrix} x_{1,1,1} & \dots & x_{1,M,1} \\ \vdots & \vdots & \vdots \\ x_{L,1,1} & \dots & x_{L,M,1} \end{bmatrix} \ddots \\ \ddots \begin{bmatrix} x_{1,1,N} & \dots & x_{1,M,N} \\ \vdots & \vdots & \vdots \\ x_{L,1,N} & \dots & x_{L,M,N} \end{bmatrix} \end{matrix}$$

Scalars, Vectors, Matrices, Tensors

- Scalar, x, just a number. **0th order tensor.**
- Vector, x, of length L
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  1th order tensor.
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  2nd order tensor.
- Tensor, x of size $L \times M \times N$
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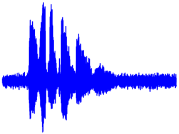
  3rd order tensor.

How do we represent signals as these?

- Sounds, Time Series

$$x^\top = \begin{bmatrix} x_1 & \dots & x_L \end{bmatrix} = [\quad]$$



- Images

$$X = \begin{bmatrix} x_{1,1}, & \dots & x_{1,M} \\ \vdots & \vdots & \vdots \\ x_{L,1} & \dots & x_{L,M} \end{bmatrix} = [\quad]$$

- Videos as tensors.. and so on..

Table of Contents

Index/Array Notation

- We need good ways to communicate operations on these objects.
- **Option 1:** Index Notation
  - Micro-level and detailed, but not very compact
- **Option 2:** Array Notation
  - Compact but abstracts away the details

Index Notation

- We define the elements in index form.
  - Element-wise multiplication:

$$c_i = a_i b_i$$

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$$C_{ik} = \sum_j A_{ij} B_{jk}$$

  - Some random tensor operations

$$C_{im} = \sum_{j,l,k} A_{ijlk} B_{mjlk}, \quad c = \sum_{i,j} A_{ij} B_{ij}$$

Array Notation

- We define the elements in index form.
  - Element-wise multiplication:

$$c = a \odot b, \;\; c \in \mathbb{R}^L$$

  - Inner product of vectors

$$c =< a, b >= a^\top b, \;\; c \in \mathbb{R}$$

  - Outer product of vectors

$$c = a \otimes b = ab^\top, \;\; c \in \mathbb{R}^{L \times M}$$

  - Matrix-vector product

$$c = Ab, \;\; c \in \mathbb{R}^L$$

  - Matrix multiplication

$$C = AB, \;\; C \in \mathbb{R}^{L \times M}$$

  - Some random tensor operations

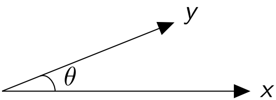
$$C = A \times_{jlk} B, \;\; C \in \mathbb{R}^{L \times M} \;\; c = A \times_{i,j} B, \;\; c \in \mathbb{R}$$

Index vs Array Notation

- Index Notation is very specific, not ambigous
- But the array notation makes it possible to manipulate the operations with ease. (E.g. gradient calculations)

The dot product

- $c = \sum_i a_i b_i = a^\top b = \|a\| \|b\| cos\theta$



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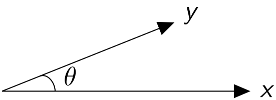

- Note that,

$$\theta = \arccos \left(\frac{a^\top b}{\|a\| \|b\|} \right)$$

- So, dot product is a great tool to measure similarity.

Matrix-Vector Product

- $c = Ab$, or $c_i =< A_{i,:}, c >= \sum_j A_{ij} c_j$. A is a matrix, b is vector. c is a what?

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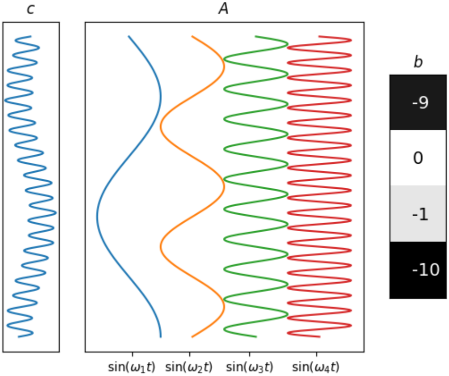

Matrix-Vector Product - 2nd interpretation

- It's a series of dot products.
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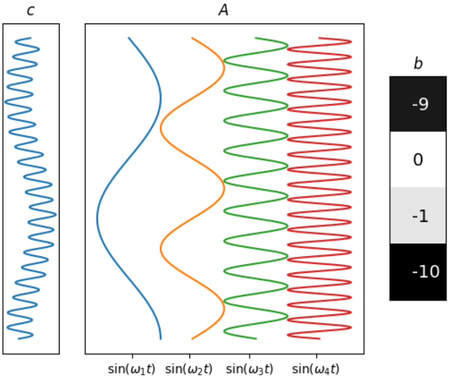

Matrix-Matrix Product

- It's a series of Matrix-vector products. (or series of inner products on a grid)
- $C = AB$, or $C_{ij} = \sum_k A_{ik} C_{kj}$, or $C_{ij} = A_{i,:}^\top C_{:,j}$
- $$C = \begin{bmatrix} A_{1,:}^\top \\ A_{2,:}^\top \\ A_{3,:}^\top \end{bmatrix} \begin{bmatrix} B_{:,1} & B_{:,2} & B_{:,3} \end{bmatrix} = \begin{bmatrix} A_1^\top B_1 & A_1^\top B_2 & A_1^\top B_3 \\ A_2^\top B_1 & A_2^\top B_2 & A_2^\top B_3 \\ A_3^\top B_1 & A_3^\top B_2 & A_3^\top B_3 \end{bmatrix}$$

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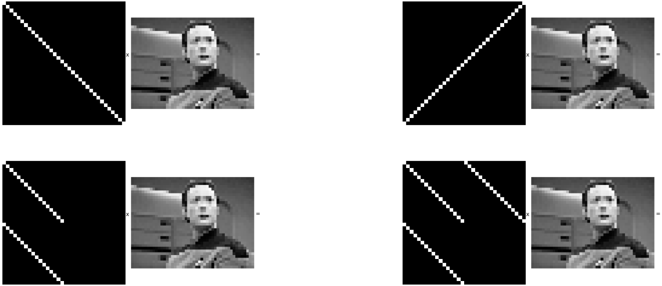
- It's a series of Matrix-vector products. (or series of inner products on a grid)
- $C = AB$, or $C_{ij} = \sum_k A_{ik} C_{kj}$, or $C_{ij} = A_{i,:}^\top C_{:,j}$
- $$C = \begin{bmatrix} A_{1,:}^\top \\ A_{2,:}^\top \\ A_{3,:}^\top \end{bmatrix} \begin{bmatrix} B_{:,1} & B_{:,2} & B_{:,3} \end{bmatrix} = \begin{bmatrix} A_1^\top B_1 & A_1^\top B_2 & A_1^\top B_3 \\ A_2^\top B_1 & A_2^\top B_2 & A_2^\top B_3 \\ A_3^\top B_1 & A_3^\top B_2 & A_3^\top B_3 \end{bmatrix}$$
- Not any pair of two matrices can be multiplied. You need to have equal number of columns from A, number rows from B.
- Master this, it will help! This has to become muscle memory.

Visualize the matrix product



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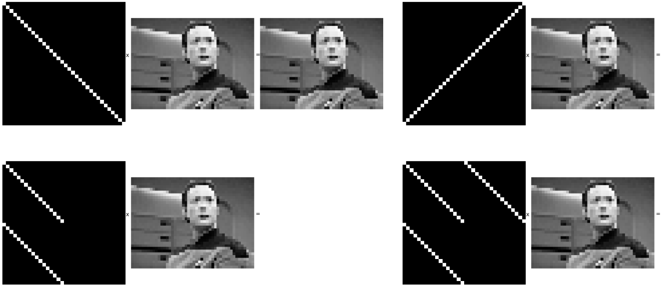

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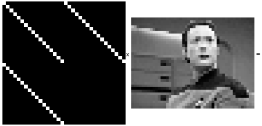

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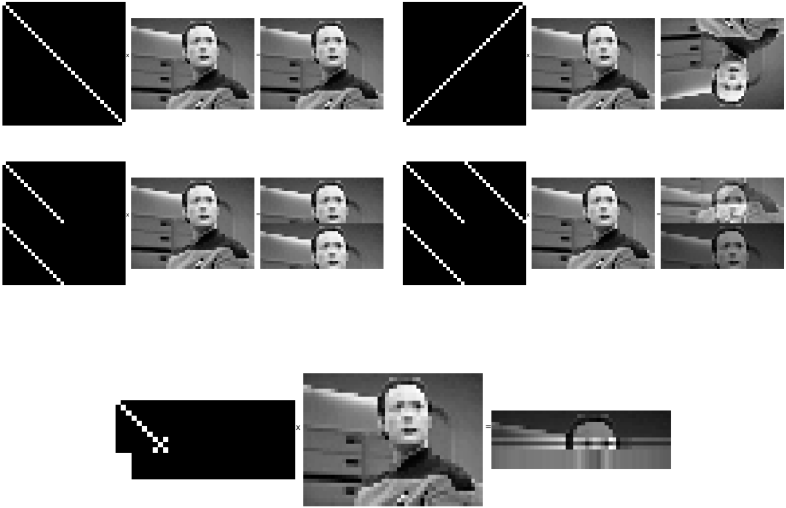
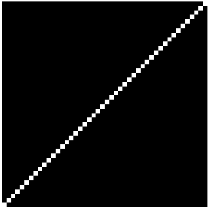


Multiplying from the other side

x



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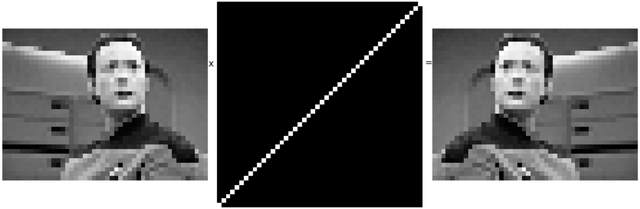

Reversing on the horizontal axis

Einstein Notation

- Let's go beyond matrices!
- $C_{i,j} = \sum_{l,k} A_{i,l,k} B_{l,j,k}$

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Let's do more Einstein stuff

- Element-wise multiplication:

$$c = a \odot b,\ c \in \mathbb{R}^L$$

- Inner product of vectors

$$c =< a, b >= a^\top b,\ c \in \mathbb{R}$$

- Outer product of vectors

$$c = a \otimes b = ab^\top,\ c \in \mathbb{R}^{L \times M}$$

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$$C = AB, \; C \in \mathbb{R}^{L \times M}$$

- Some random tensor operation

$$C = A \times_{jlk} B, \; C \in \mathbb{R}^{L \times M} \quad c = A \times_{i,j} B$$

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$$C = A \times_{jlk} B,\ C \in \mathbb{R}^{L \times M}\ \ c = A \times_{i,j} B$$

$$A_{ijlk}, B_{mjlk} \rightarrow C_{im}$$

Implementing Einstein products is easy in Python

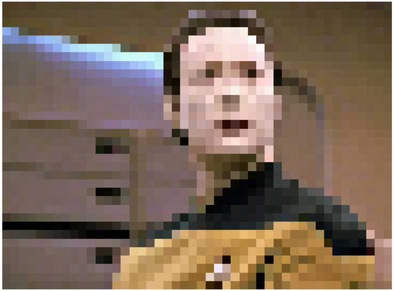
- Batch Matrix Multiplication

$$A_{bij}B_{bjk} \to C_{bik}$$

```
C = torch.einsum('bij,bjk->bik', A, B)
```

Application of Tensor Operations

- RGB images



- Let us apply a matrix multiplication to each channel, and then average over the channels.

Application of Tensor Operations

- In Index Notation

$$C_{ij} = \sum_{k,c} \underbrace{B_{ik}}_{\text{Matrix}} \underbrace{A_{kjc}}_{\text{image}} \underbrace{w_c}_{\text{WtOverCh.}}$$

- Notice that this notation can handle multilinear operations.

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- In Einstein Notation:

$$B_{ik}, A_{kjc}, w_c \to C_{ij}$$

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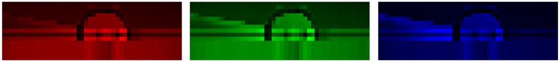
Application of Tensor Operations

- First step

$$B_{ik}, A_{kjc} \rightarrow T_{ijc}$$



- Second step

$$T_{ijc} w_c \rightarrow C_{ij}$$

Let's also see some reshaping operations

- Vectorization:

$$\text{vec}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

- The 'Diag' Operation:

$$\text{Diag}\left(\begin{bmatrix} a_1 & a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

- The 'Reshape' Operation:

$$\text{Reshape}_{32}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{22} \\ a_{21} & a_{13} \\ a_{12} & a_{23} \end{bmatrix}$$

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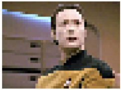
- Let's visualize this,

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \otimes$$



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- The matrix form could be helpful when calculating gradients, and coming up with efficient implementations.
- Einsum is not as optimized as matrix multiplication.

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Matrix inverse

- Let's think about a linear system,

$$\begin{aligned} Ax =& b \\ \rightarrow A^{-1}Ax =& x = A^{-1}b \end{aligned}$$

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- Is A^{-1} always defined?
- First, A needs to be square.
- Second, it needs to be full rank. Columns of A need to be linearly independent.

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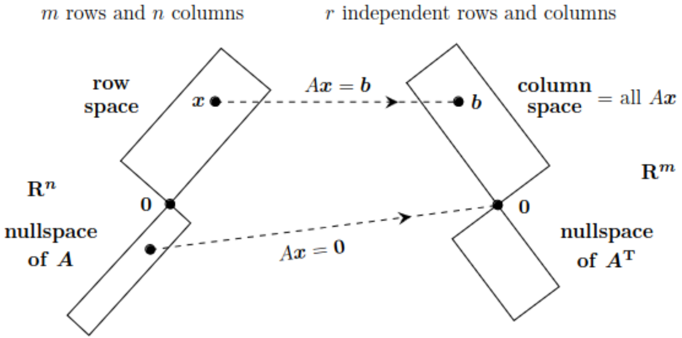
- $A^\dagger := (A^\top A)^{-1} A^\top$. This is known as the pseudo inverse.
- This is essentially least squares. (We will show that later)

Four Fundamental Subspaces in Linear Algebra



BIG PICTURE OF LINEAR ALGEBRA

row space ⊥ nullspace   column space of A ⊥ nullspace of A^T

row rank = column rank = r

Image Taken from Gilbert Strang's 'Introduction to Linear Algebra' book.

Norms, trace

- l_2 norm: $\|x\|_2 = \sqrt{\sum_j x_j^2}$. Also known as Euclidean Norm.
- l_1 norm: $\|x\|_1 = \sum_j |x_j|$.
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- Frobenius norm: $\|X\|_F = \sqrt{\sum_i \sum_j |X_{ij}|^2} = \sqrt{\mathrm{tr}(XX^\top)}$

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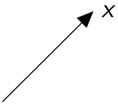
- We are just giving an idea here with simple examples. We will see these more in real action later. (hint: backprop)

Table of Contents

Eigenvalues / Eigenvectors

- $Ax = \lambda x$



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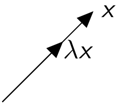

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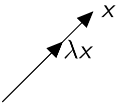

- Note that x doesn't change its direction.
- Eigenvectors are 'characteristic' directions for the system described by A.

Finding the Eigenvectors

- The 'Linear Algebra Class Way':
- Let's have this matrix

$$A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$$

- Calculate the determinant (why?)

$$\det(A - \lambda I) = \begin{vmatrix} 0.8 - \lambda & 0.4 \\ 0.2 & 0.6 - \lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.40$$

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- Then we find vectors in the null space of $A - \lambda I$
- $A - I = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix} v = 0$, find a non-zero vector v such that the equation is satisfied. $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Finding the Eigenvectors

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- To get all the eigenvectors we can deflate the matrix. Just subtract v, and repeat the process..

Ok, but how is this a decomposition?

- $AV = V\Lambda$, where columns of V are the eigenvectors, and Λ is a diagonal matrix with eigenvalues on the diagonal.

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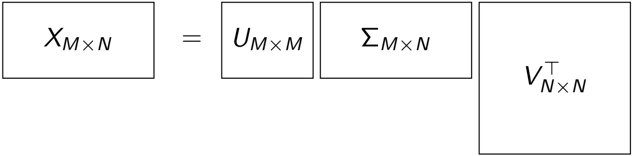
- $AV = V\Lambda$, where columns of V are the eigenvectors, and Λ is a diagonal matrix with eigenvalues on the diagonal.
- And here's the decomposition $A = V\Lambda V^{-1}$.
- But notice that this decomposition is only defined for square matrices.

Singular Value Decomposition

- Let us given a matrix of size X in $\mathbb{R}^{M\times N}$.
- $X = U\Sigma V^\top$, $U \in \mathbb{R}^{M\times M}$ and is orthogonal $U^\top U = I$, $\Sigma \in \mathbb{R}^{M\times N}$ is a matrix with non-zero elements on the main diagonal, and $V \in \mathbb{R}^{N\times N}$, and is orthonal $VV^\top = I$.



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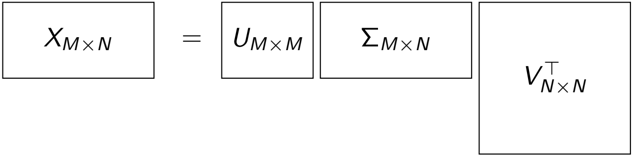
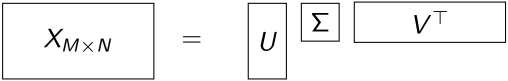


- An alternative way of viewing it is $X = \sum_{k=1}^{M} \sigma_k u_k v_k^\top$. Note that we can cut the sum short, and keep the biggest singular values! (set $X = \sum_{k=1}^{K} \sigma_k u_k v_k^\top$, $K \leq M$)



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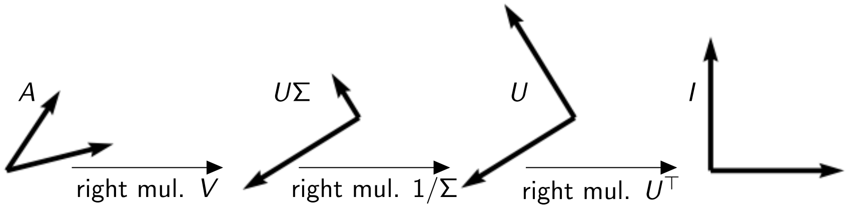
- For positive semi-definite matrices, SVD and eigenvalue decomposition are equivalent.

Geometric Interpretation of SVD



List of Decompositions

- **LU decomposition:** $X = LU$, L is lower triangular, U is upper triangular.
- **QR decomposition:** $X = QR$, Q is a matrix with orthonormal columns, R is an upper triangular matrix.
- **Eigenvalue decomposition:** $X = U\Lambda U^{-1}$, columns of U are eigenvalues of X, which is square (diagonalizable) matrix.
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- There's more, e.g. Cholesky, NMF, CR, ICA, ...

List of special type of matrices we'll see in this class

- **Rotation matrices**
- **Markov matrices** (Probability Transition Matrices)
- **Transform matrices** (Fourier Transform, Convolution,...)
- **Covariance matrices** (Define a Multivariable Random Variable)
- **Adjacency matrices** (Define a Graph)

Recap

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- We took a glimpse into how we can decompose signals.
- We gave a crude summary into what we need from Linear Algebra.

Recommended Reading

- Gilbert Strang, Introduction to Linear Algebra, https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/video_galleries/video-lectures/, https://math.mit.edu/~gs/linearalgebra/ila5/indexila5.html
- Trefethen and Bau, Numerical Linear Algebra, https://people.maths.ox.ac.uk/trefethen/text.html
- Matrix Cookbook, http://www2.imm.dtu.dk/pubdb/edoc/imm3274.pdf

What's Next

- Probability Calculus, Random Variables, Multi-dimensional Distributions
- Exponential Family Distributions
- Maximum Likelihood, MAP, Bayesian parameter estimation principles
- Labs are starting next week! (first one is Sept. 15)